NEDERL. AKADEMIE VAN WETENSCHAPPEN

# PROCEEDINGS

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President: J. VAN DER HOEVE

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## CONTENTS

**Physics.** — *Les déformations plastiques près des entailles.* Par F. K. TH. VAN ITERSON.

(Communicated at the meeting of January 31, 1942.)

§ 1. *Introduction.*

L'acier doux à environ un dixième pourcent de carbone constitue le métal usuel employé pour la construction des navires, des charpentes metalliques, des chaudières, etc. Ce métal doit son aptitude à l'exécution de ces ouvrages d'art à sa ductilité comme à sa grande résistance à la rupture.

Le calcul des tensions dans nos constructions, sollicitées par des charges extérieures, des différences de températures, des retraits de soudure, etc. révèle de fortes concentrations de forces internes près des angles rentrants.

On s'est rendu vaguement compte que pour les charges statiques la ductilité atténue la concentration des tensions à ces endroits, par exemple à la périphérie de la section du raccord entre la tige et la tête d'un boulon. Mais pour faire entrer la ductilité dans nos calculs de résistance, il faut commencer par mettre en formules la distribution des tensions ainsi que les glissements et déformations autour des entailles.

Les lois physiques qui commandent ces calculs sont traitées dans les manuels d'élasticité. Nous nous référons à ce sujet au Handbuch der Physik, Band VI. Mechanik der elastischen Körper, Kapitel 6, Plastizität und Erddruck de A. NADAI, 23. Das ebene Problem des Gleichgewichts vollkommen plastischer Massen, p. 472.

Ainsi nous basons nos calculs sur les lois de la plasticité formulées par DE SAINT VÉNANT, mais avant d'entrer dans le sujet nous commençons par la description d'un cas de plasticité que nous avions résolu il y a trente ans [1]), d'abord rejeté par les professionnels [2]) et maintenant généralement accepté [3]).

§ 2. *Les glissements plastiques dans les parois épaisses de cylindres.*

Pour la matière parfaitement plastique on accepte que les glissements, les déformations, se produisent aux endroits où la tension de glissement atteint la limite $k$, ainsi $\tau_{max} = k$. Quand on augmente la pression intérieure dans un cylindre de matière plastique cette tension dans la paroi est d'abord atteinte à l'intérieur et puis se propage; des surfaces de glissement se développent dans la masse plastique et se propagent vers l'extérieur. Quand le glissement atteint l'extérieur du cylindre, celui-ci commence à se gonfler et quand on ne diminue pas la pression, il se crève.

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[1]) Engineering, Jan. 5, 1912, p. 22, F. VAN ITERSON, The Strength of Thick Hollow Cylinders.

[2]) Engineering, Jan. 12, 1912, p. 58, COOK et ROBERTSON répondirent: „The agreement of the experimental values with those calculated from the above formula is remarkable but must nevertheless be accidental".

[3]) VON KÁRMÁN, Ueber elastische Grenzzustände. Verhandlungen des internationalen Kongresses für technische Mechanik, Zürich, 1926.

Handbuch der Physik, Band VI, 1928. Mechanik der elastischen Körper. Das ebene Problem des Gleichgewichtes vollkommen plastischer Massen, p. 474.

TIMOSHENKO, Strength of Materials, Part II, p. 528.

NÁDAI, Plasticity, 1931, p. 186 et 227. The thick-walled tube under internal pressure.

HÜTTE, 26. Auflage, I, 1936, IV, Mechanik der bildsamen Körper, p. 347, Dickwandiges Rohr.

La pression extrême, intérieure ou extérieure, que peut supporter un cylindre ductile est $p = 2\,k\,ln\,\frac{r_e}{r_i}$ où $r_e$ = rayon extérieur, $r_i$ = rayon intérieur.

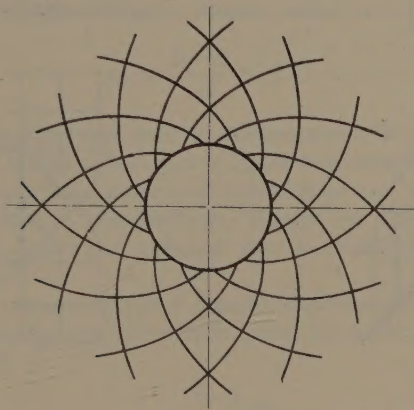

Fig. 1.
Lignes de glissement dans la zone de déformation plastique autour d'un trou cylindrique à charge intérieure ou extérieure de la masse.

Dans la figure 1 nous représentons les lignes de glissement autour d'un trou cylindrique dans la masse plastique soumise à une pression intérieure ou extérieure. Ces lignes forment deux faiscaux de spirales logarithmiques qui s'entrecoupent orthogonalement.

Quand on progresse le long d'une ligne les deux tensions principales $\varrho_r$ et $\varrho_t$ s'accroissent ou diminuent toutes deux de $2\,k\,\varphi$, où $\varphi$ représente l'angle dont la tangente a tourné. La différence entre les deux tensions principales reste constante $\varrho_t - \varrho_r = 2\,k$ (thèses de HENCKY).

§ 3. *Les glissements et les tensions auprès des angles d'un trou carré.*

Imaginons un cylindre muni d'un trou central chargé par pression interne.

Le calcul selon les lois de l'élasticité [1]) nous révèle des tensions outre mesure auprès des angles vifs rentrants, mais nous savons qu'en réalité la ductilité du métal empêche un accroissement excessif des tensions. Pour se rendre compte de ce qui se passe autour d'un trou carré quand on charge l'élément de construction, il faut donc recourir à la théorie de la plasticité. La question se pose de résoudre deux équations différentielles partielles simultanées. Mais le problème se simplifie beaucoup par le fait que quand on prend le sommet de l'angle comme origine toutes les quantités restent constantes pour un rayon et sont uniquement fonction d'une seule variable à savoir, l'angle que fait ce rayon avec la ligne de symétrie. Et cependant quand on prend à tâche de trouver une solution continue, satisfaisant aux conditions des deux limites données et de la symétrie, on sent que la solution doit être très simple, mais la difficulté de la trouver fait renoncer à tout espoir. Mais du moment que dans sa recherche on abandonne la condition de continuité et qu'on accepte que le champ de déformation plastique se subdivise en zones contiguës, l'analyse du problème se présente toute simple, non seulement pour le trou carré, mais aussi pour les autres cas d'entailles rectilignes.

Dans la suite nous donnons peu de texte et nous décrivons les champs de lignes de glissement desquels on déduit la répartition des tensions d'après la thèse de HENCKY.

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[1]) C. E. INGLIS, Transactions of the Institution of Naval Architects, 1911, Part I.
C. B. BIEZENO und R. GRAMMEL, Technische Dynamik, 1939.
H. NEUBER, Kerbspannungslehre, 1937.

Lignes de glissement auprès des angles d'un trou carré (Figure 2). On y distingue trois zones. Dans 1 et 3 l'état des tensions ne varie pas, mais dans le secteur 2 les tensions augmentent proportionnellement à l'angle du rayon. Dans la ligne de symétrie on a $\varrho_1 = 2k\left(1 + \frac{\pi}{4}\right) - p$ comme tension principale maximum.

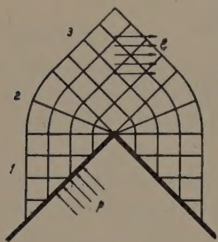


Fig. 2.

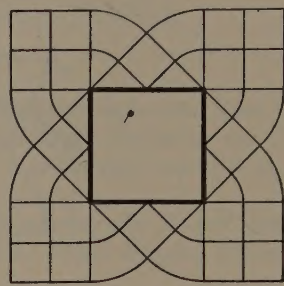


Fig. 3.

Les déformations plastiques commencent près des angles quand on charge un élément de construction comportant un trou de section carrée par des tensions intérieures ou extérieures (Figure 4).

Pour le coin gauche en bas nous avons indiqué la zone et les lignes de glissement pour un angle arrondi. Ces lignes sont des spirales logarithmiques s'entrecoupant sous des angles de 90°.

Au moment où l'écoulement de la masse plastique s'est propagé sur toute la longueur des côtés du trou on obtient la figure 3 pour les lignes de glissement. Aux limites de la

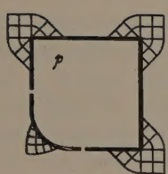


Fig. 4.

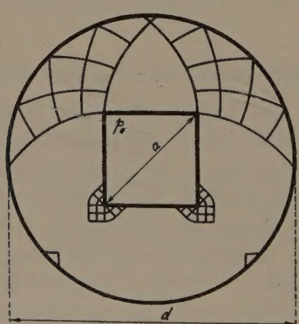


Fig. 5.

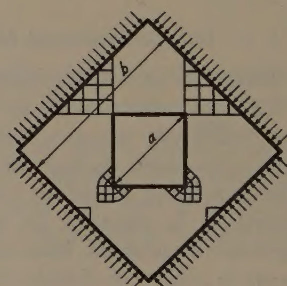


Fig. 6.

figure la déformation plastique est nulle. Ici et en dehors il n'y a que des déformations élastiques.

La pression intérieure ou extérieure extrême que peut supporter un cylindre de diamètre $d$ percé d'un trou carré de diagonales $a$ (figure 5) est

$$p_e = 2k \ln \frac{d}{a}$$

où $k$ est la tension maximum de cisaillement qui fait glisser la matière parfaitement plastique.

Le commencement de déformation plastique est indiqué en bas dans la figure. Quand une certaine pression dépendant de la proportion entre la résistance au cisaillement $k$ et le

coefficient d'élasticité $E$ est dépassée le glissement selon les spirales logarithmiques sortant de la périphérie prend le dessus et se propage vers les coins jusqu'à ce que les zones de glissement indiquées en haut dans la figure se sont développées.

La pression extrême que peut supporter un tube de section carré de matière ductile avec trou carré placé de biais (figure 6) est

$$p_e = \frac{b-a}{a} \times 2k.$$

Les zones et lignes de glissement pour le commencement de la déformation plastique sont indiquées en bas de la figure. Les zones et lignes de glissement, qui prennent le dessus quand on pousse plus loin la déformation, sont indiquées en haut.

Barre ou plaque percée d'un trou conforme, chargée par pression extérieure ou intérieure. Les lignes de glissement sont indiquées pour le commencement de la déformation plastique (figure 7).

Tube carré avec paroi épaisse soumis à pression extérieure ou intérieure jusqu'à glissement de la matière plastique à travers toute la paroi (figure 8).

Pression extrême

$$p_e = \frac{t-s}{s}\, 2k.$$

La déformation commence près des angles rentrants, puis cette déformation cesse et une autre déformation se développe partant du milieu des côtés, indiquée dans cette figure et qui mène à la rupture.

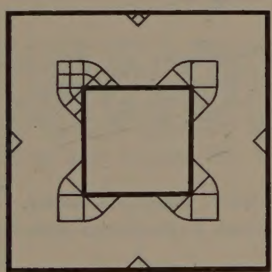

Fig. 7.

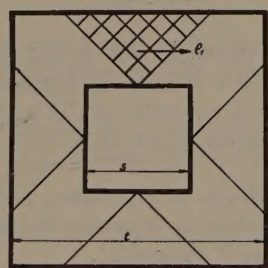

Fig. 8.

§ 4. *Les lignes de glissement et les tensions autour de trous de différentes formes et de fentes dans la masse plastique.*

La tension principale dans le champ de déformation plastique au dessus de l'angle d'un trou triangulaire équilatéral (figure 9) dans une masse soumise à une pression extérieure uniforme est

$$\varrho_1 = 2k\,(1 + \tfrac{7}{12}\pi).$$

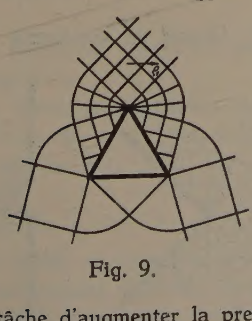

Fig. 9.

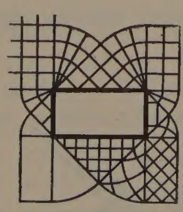

Fig. 10.

Quand on tâche d'augmenter la pression le trou se ferme et les lignes de glissement des carrés s'allongent.

Trou rectangulaire (figure 10).

Trou rectangulaire.

Il est instructif de construire les lignes de glissement autour d'un trou rectangulaire. Quand on continue à étendre ou à comprimer la matière plastique les rectangles remplis de lignes de glissement s'agrandissent comme indiqué à gauche en haut.

En bas à droite nous avons dessiné les trajectoires des tensions qui entrecoupent les lignes de glissement sous des angles de 45°.

Fente (figure 11).

Dès qu'une pièce de matière ductile munie d'une fente est étirée perpendiculairement à celle-ci dans tous les sens, il se développe un champ de déformation plastique près des bouts de la fente. Chaque champ consiste en 5 zones, deux triangles rectangles, deux secteurs de 90° et un carré.

Fente (figure 12).

Quand on tire devantage, d'abord le champ de déformation plastique se développe comme indiqué en lignes tracées, puis les carrés s'agrandissent, de la manière indiquée à gauche en lignes pointillées, mais on ne saurait déterminer exactement jusqu'où s'étend la

Fig. 11.

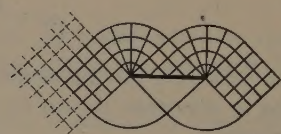

Fig. 12.

déformation plastique, ceci dépend des déformations élastiques qui pour le présent ne peuvent pas être calculées. Il faut comparer ce cas de déformation plastique avec l'étude classique sur la plasticité par le maître L. PRANDTL, Proc. of the 1. intern. congr. of applied mechanics, p. 43, Delft 1925.

§ 5. *Le champ de déformation plastique pour les barres entaillées étirées.*

La masse plastique commence à couler près des pointes d'entaille au plus léger étirage par des forces $P$ (figure 13).

Quand $P = 2k(1+\vartheta)b$ on est arrivé au maximum que peut supporter la section rétrécie, du moins quand on ne considère que le problème à deux dimensions, p.e. une barre très épaisse dans le sens perpendiculaire à la feuille de dessin (figure 14). Pour un couple fléchissant on obtient les mêmes lignes de glissement.

Développé à plein le champ de déformation plastique s'étend sur la surface indiquée à côté et quand on étire davantage la taille se rétrécit et la figure de déformation diminue. On reconnaît l'analogie de ce cas de déformation plastique avec le problème traité par PRANDTL de l'angle obtus pressé au bout. Quel moment de flexion peut transmettre la section dangereuse?

Barre avec deux entailles opposées (figure 15). Quand on commence à étirer cette

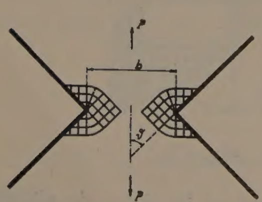


Fig. 13.

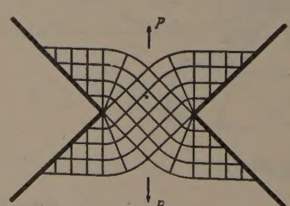


Fig. 14.

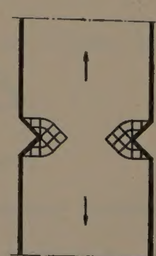

Fig. 15.

barre des zones de déformation plastique se développent près des entailles. L'étendue de ces zones dépend de l'élasticité de la matière.

Quand on continue l'étirage de la barre constituée de matière parfaitement plastique, brusquement la tension principale diminue et il se produit la zone de lignes de glissement indiquée dans figure 16.

Pour se donner une idée de la contraction ou extension latérale près des entailles de la figure précédente on commence par étudier le cas des entailles élongées (figure 17). Cette figure résulte de la loi de HENCKY et la figure 16 en est déduite.

$$\varrho_1 = 2k.$$

Au début de l'étirage d'une barre à deux incisions (figure 18), des zones de déformation plastique se développent près des fonds des incisions. Quand on continue à étirer la barre

Fig. 16.

Fig. 17.

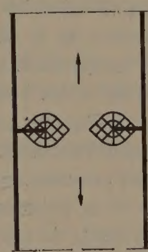

Fig. 18.

il se produit brusquement la zone de déformation identique à celle de la figure 16.

Il est à noter que toutes ces figures, la figure 13 et suivantes, peuvent aussi servir pour des barres soumises à la flexion par un couple.

Champ des lignes de glissement dans une barre à traction avec trou carré relativement grand, placé de biais (figure 19); à gauche début de la déformation plastique, à droite terminaison.

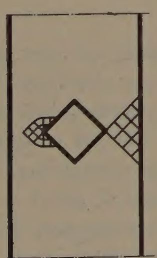

Fig. 19.

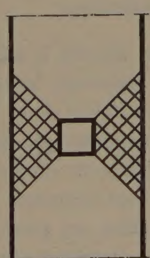

Fig. 20.

Fig. 21.

Champ des lignes de glissement dans une barre à traction avec trou carré placé d'aplomb (figure 20).

Champ des lignes de glissement dans une barre à traction avec trou cylindrique (figure 21). A gauche commencement de la déformation, à droite glissement sur toute l'épaisseur à côté du trou.

§ 6. *Conclusions.*

Nous avons tâché de controler par des essais, par pression intérieure de cylindres à parois épaisses percées d'un trou carré central et par des essais à la traction de barres d'acier doux entaillées ou munies d'un trou carré transversal, la résistance à la déformation plastique et les lignes de glissement déduites de la théorie. M. FOKKINGA, ingénieur des

Mines de l'Etat néerlandaises, spécialisé en métallographie, a exécuté ces essais avec beaucoup de soins. Quelques uns sont faits dans le laboratoire VAN DER WAALS de l'université d'Amsterdam sous la direction du Prof. MICHELS.

Les lignes de glissement rendues visibles au moyen de la liqueur „Fry" semblent confirmer la théorie mais les essais menant à la rupture ont révélé une certaine divergence.

Pour les cylindres à trou carré (figure 5) la formule $p_i = 2\,k\,ln\,\frac{d}{a}$ où $2\,k$ représente la résistance à la traction a été bien confirmée pour l'acier doux recuit d'une résistance à la traction de 40 kg/mm², jusqu'à la proportion $d : a = 2$. Pour les plus grands diamètres la résistance du cylindre augmente au-dessus de celle indiquée par la formule théorique, jusqu'à atteindre 1,44 la valeur calculée pour une proportion $d : a = 4$ ($d = 55$ mm $a = 14$ mm).

En poussant davantage la proportion $d : a$ la différence devient moins prononcée. Pour $d : a = 5$ ($d = 70$ mm $a = 14$ mm) la résistance est encore 1,25 fois celle calculée.

L'explication de la divergence est celle-ci:

Il n'y a concordance que quand la paroi peut se contracter en épaisseur d'une manière similaire à celle des barres de comparaison soumises à des essais de traction.

Pour les épaisseurs en dessus de $d : a = 2$ on observe que la contraction ne peut pas se développer librement. Et quand $d : a$ se rapproche de 5 ou surpasse cette proportion les cylindres commencent à se déchirer dans les coins du trou, sans contraction appréciable des parois et la pression intérieure $p$ s'exerce sur un diamètre plus grand que $a$.

Pour les essais de traction sur barres entaillées la différence observée dans la résistance avec les barres lisses s'explique de même par empêchement de la contraction.

Aussi très souvent les constructions métalliques se sont rompues aux endroits où la plasticité devait atténuer les tensions.

Pour la pratique il est d'une importance particulière de connaître la cause de la divergence entre la théorie et la réalité. Pour pouvoir la comprendre il faut introduire une nouvelle conception dans la théorie de la plasticité, celle de la déformation spécifique. Un élément carré dans le champ de déformation plastique orienté selon les tensions principales de côté $a$ devient un rectangle de côtés $a + \triangle a$ et $a - \triangle a$ après déformation. Nous appelons $\frac{2 \triangle a}{a} = s$ la déformation spécifique.

Quelques matières plastiques, le fer chauffé à mille degrés dans son état austénitique, certaines résines et masses plastiques modernes, le verre amolli par chauffage, approchent plus ou moins de la matière idéale, mais l'acier doux recuit ne supporte qu'une déformation spécifique très restreinte. En effet lorsqu'on fait un essai de traction et quand l'effort auquel est soumis le métal atteint la charge de rupture, on voit un étranglement se dessiner en un point de la barre, étranglement qui augmente jusqu'à ce que le métal se brise dans sa section la plus contractée. La striction est pour l'acier de construction de l'ordre de 50 %. Pour une matière parfaitement plastique elle devait être 100 % et la charge de rupture devait donc se réduire à zéro.

Quand on exprime en chiffres la déformation spécifique $s$ on trouve qu'elle devient très grande, même infiniment grande près de l'extrémité d'une fente.

La partie de la section voisine de la fente atteindra la déformation de rupture bien avant que le reste de la section ait pris la déformation maximum qu'elle aurait pu supporter.

Dans le fer la rupture commencera donc près de la fente et se propagera dans toute la section de proche en proche et pour les mêmes raisons.

Il est évident que ce qui retardera la rupture, ce sera la faculté qu'aura le métal, de prendre sans se rompre, un allongement considérable, mais on ne peut pas compter sur la ductilité du fer à un degré tel que le suppose la théorie de la déformation des matières de plasticité absolue. Un défaut local, une fente, si petite qu'elle soit, peut devenir pour l'acier le point de départ d'une rupture transversale.

Une autre propriété du fer est cause que les résultats des essais diffèrent de ceux prédits par la théorie. Celle-ci suppose que la tension de cisaillement, la résistance au glissement, reste constante pendant les déformations plastiques, $\tau = k$, mais pour le fer $k$ augmente avec la déformation spécifique $s$ dans une mesure considérable; au moment de la rupture elle est à peu près doublée. C'est une circonstance favorable.

Mais le fer a une autre propriété qui le rend inférieur aux matières plastiques vraies.

La répétition des efforts, surtout les renversements de sens des efforts, est pour les métaux une cause spéciale d'altération [1]).

Nous avons vu que tout près de la pointe de l'angle vif le métal infailliblement se déforme par la charge, mais cette déformation plastique est généralement localisée dans une zone très restrainte et la déformation est élastique sur la plus grande partie de la section. Quand la pièce de construction est déchargée, la zone déformée ne s'ajuste plus et est écrouie par le retrait de la pièce. Un commencement de fissure se produit entre les cristaux du métal qui bientôt devient fatal si les chargements et déchargements de la construction se répètent.

On serait tenté d'appliquer la théorie de la plasticité pour le calcul des constructions métalliques. A la fin de cette étude, l'auteur ne peut s'abstenir de donner l'avertissement de pratiquer beaucoup de modération à ce sujet.

M. A. HELLEMANS ingénieur physicien et électricien a bien voulu discuter avec nous le sujet de cet article.

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[1]) „Een geval van kerfwerking" par F. K. TH. VAN ITERSON. De Ingenieur, 1938, No. 40, Werktuig- en Scheepsbouw 7.

**Geophysics.** — *Topography and Gravity in the North Atlantic Ocean.* By F. A. VENING MEINESZ.

(Communicated at the meeting of January 31, 1942.)

For the investigation of the Earth's crust under the oceans our data are scanty; we have only three sources of information. In the first place we can now obtain a detailed knowledge of the topography of the sea-bottom thanks to the new method of sonic sounding which so much reduces the trouble of determining the sea-depth. In the second place we have the data given by dredging, eventually made more valuable by shooting tubes over a few meters in the sub-oceanic soil. Thirdly we may obtain gravimetric results. It is true that we can also make a magnetic survey of the oceans as it has e.g. been done by the famous cruises of the ship „Carnegie" of the CARNEGIE Institution of Washington, but these results, though of high importance for the study of terrestrial magnetism, do not in general give much clear information about the crust and so we shall not further mention them here. Besides we of course also dispose of the geological evidence obtained on the oceanic islands and near to the coasts.

In this paper we shall give a provisional study especially of the submarine topography and the gravity results over part of the North Atlantic, i.e. over the Azores Archipelago and between the Azores and Europe. As the chart shows we dispose here of a fairly large number of gravity observations which, as it is well-known, the writer has been able to make on board of submarines of the Royal Dutch Navy; he feels a great debt of gratitude for the many opportunities given to him. He likewise feels indebted to the Netherlands Geodetic Commission on whose behalf the expeditions have been organized and which has also defrayed the expenses for the great amount of computational work needed to obtain the results published here.

The area is also well covered with soundings. The accompanying map showing the contour lines is a copy of a larger one made by Mr. BLOEM of the Hydrographic Service in the Hague for the report of the Netherlands Geodetic Commission about the gravity results obtained at sea [1]). It has been derived from the chart of the North Atlantic of the "Carte Bathymétrique des Océans", 3rd edition, issued by the International Hydrographic Bureau in Monaco in 1935, supplemented by the charts of the Azores and of the Altair area published by A. DEFANT and G. WÜST [2]).

Nevertheless, although the soundings and the gravity data are numerous compared with other oceanic areas, much more will be needed before a detailed study of this area can be made. Still it is worth while to examine the data now available; we shall see that we can already draw some conclusions and make some surmises.

Beginning by the topography we see that the map clearly shows a linear arrangement of most of the features. This is especially clear in the Azores where it has already been often remarked that in the topography two directions are predominant, one from WNW to ESE, i.e. under an azimuth of about 65° west, and the second from NE to SW, i.e. under an azimuth of about 45° east. Nearly all the islands and the submarine elevations have their length-axis in the first sense except the western group of islands of Flores and Corvo where the ridge follows the main direction of the Mid Atlantic Rise; they form in fact part of this rise, which here follows more or less the second direction. The middle

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[1]) F. A. VENING MEINESZ, Gravity Expeditions at Sea, Vol. IV, to be issued in this year or the next.

[2]) A. DEFANT und BJ. HELLAND–HANSEN, Bericht über die ozeanographischen Untersuchungen im zentralen und östlichen Teil des Nordatlantischen Ozeans; A. DEFANT, Die Altair Kuppe; G. WÜST, Das submarine Relief bei den Azoren, Abh. Preuss. Akad. d. W. 1939, Phys. Math. Kl, 5.

group of islands of Fayal, Pico, São Jorge, Graciosa and Terceira together with the Princess Alice Bank in its general grouping also shows the second direction and this is likewise more or less true for the eastern group of São Miguel, Santa Maria and the bank to the south of it.

As far as the writer knows it has been less generally realized that these two directions are not only valid in the Azores but that we can probably also trace them in the whole area of the map east of this archipelago, although here and there slightly changed in direction. We find the second direction e.g. in a long ridge to the NE of the Azores, in a ridge to the NE of Madeira running towards the Seine Bank, in a ridge to the SW of the Josephine Bank and in a ridge connecting the Gorringe Bank with the Portuguese coast. The first direction shows itself e.g. in the ridge connecting the Cruiser Bank with the Mid Atlantic rise, in the ridge running WNW wards from a bank at about 43° N and 21° W towards this rise, in the ridge connecting the Gorringe Bank with the Josephine Bank, in a ridge to the NW of Cape Finisterre and perhaps in a ridge connecting the submarine promontory at about 43° N and 12° W with the Spanish mainland. It gives the impression that in this last area both directions are turned slightly anti-clockwise.

This linear arrangement of the topographic features in the Azores and probably also in the area east of them seems to point to block-faulting and not to folding and overthrusting; these last phenomena usually occur in curved belts while the first imply a system of more or less straight lines. The shearing is nearly everywhere accompanied by volcanism along the fault-lines; the islands show numerous instances of this as e.g. the island of São Jorge which consist of a long row of eruption points. The volcanoes often also occur in the points of intersection of the two directions as e.g. shown by the three groups of the Azores and by Madeira.

The gravity results are in good harmony with our tentative conclusion that no folding has occurred in our area. Although the seismicity of a great part of it indicates movements that are still going on, there is no evidence of any belts of strong negative anomalies as have been found in the East and West Indies and in other areas where young folding orogeny has been taking place and where these belts have been interpreted as an indication that below the folding the main part of the crust has buckled downwards. So in our area the Earth's crust appears to give way by faulting and not by buckling or folding and this seems to point here to a rather thick rigid crust. If we apply to the crust the equations of the theory of elasticity as a sufficient approximation to its behaviour, we find that a horizontal compression can only bring about a buckling of the crust if its thickness does not exceed a certain limit or if it separates in layers each of limited thickness. If the thickness exceeds these limits a compressive stress can only give shearing. So we may probably conclude to the presence of a rather thick rigid crust in the area under discussion. This seems to be in good harmony with the conclusion arrived at in a recent paper [1]) on the gravity over and near the Hawaiian Archipelago and Madeira, where the writer derived a thickness of at least 25 km for the crust when consisting of one layer only.

In case of shearing along planes in two directions we may reasonably suppose the presence of a compressive stress parallel to the bisector of the angle between the directions. A few years ago BYLAARD [2]) has derived the angle the two shearing planes

[1]) F. A. VENING MEINESZ, Gravity over the Hawaiian Archipelago and over the Madeira area; conclusions about the Earth's crust, Proc. Ned. Akad. v. Wet. Amsterdam, **44**, 1 (1941).

[2]) P. P. BYLAARD, De plastische vervorming van vloeiijzer en de berekening van ijzerconstructies, De Ingenieur, 23 (1933).

P. P. BYLAARD, Théorie des déformations plastiques et locales par rapport aux anomalies de la gravitation, aux fosses océàniques, aux géosynclinaux, au volcanisme, à l'orogénie et à la géologie de l'océan pacifique occidental, Association de géodésie, rapport du congrès d'Edinbourg (1936).

may be expected to make if the shearing takes place in a plate of an elastic material after the strength limit has been passed and plasticity has set in. He found an angle of 110° involving an angle of 55° between the stress direction and each of the shearing planes. As this condition of plastic shearing is probably fulfilled in the crust, the fact that the two directions shown by the topography do indeed enclose this angle seems a good corroboration of our hypothesis. Adopting it we would find the direction of the compression of the crust to have an azimuth of about 10° west.

Such a compression does not appear unlikely. If we admit the presence of a rigid crust under the Atlantic, we may expect similar stresses in that crust to those having caused the great crustal shortening in Europe by the Alpine orogeny. These features break off at the western shores of the continent and it seems indicated to suppose that the adjoining oceanic crust has undergone a similar shortening. The difference of constitution and eventually also of thickness of the continental and the oceanic crust may well have caused a different behaviour in giving way to these stresses. As a consequence of this hypothesis we must suppose that not the entire topography is volcanic but that part of it must have originated because of the thickening of the crust by this compression. Examining the map of the central and eastern group of the Azores, we get the impression that while the volcanoes preferentially occur in the direction WNW—ESE this other topography also occurs in the second direction. Clear evidence that tectonic phenomena are present in our area besides volcanism is also given by the seismic activity in the Azores as well as over the Mid Atlantic rise and between the Azores and Europe. We find such evidence likewise in the geomorphological and geological indications for vertical movements found in many islands. Those e.g. belonging to the Azores archipelago have been tilted westwards; the eastern parts show more effects of erosion and so are evidently older than the western parts. In Santa Maria middle miocene marine limestone is found at elevations ranging from 40 m to 120 m [1]).

The interpretation of the gravity anomalies in our area is not easy. Marked features of strong anomalies do not occur except on the islands and here they disappear by applying regional isostatic reduction [2]). So we require a more detailed gravimetric survey for getting insight in the anomalies than e.g. in the East Indies where the presence of the belt of large negative anomalies allows to trace its course by means of profiles at great distances from each other. Another consequence of the smaller size of the anomalies is the greater relative effect of the errors in the isostatic reduction resulting from a defective knowledge of the submarine topography. So we need a still larger number of observations in this area before a satisfactory gravimetric study will be possible. At this moment most of our conclusions cannot be otherwise than tentative.

The whole gravity material has been subjected to the reduction by means of the new tables for regional and local isostatic reduction according to the Airy system [3]). They have been reduced for values of the crustal thickness $T$ of 20 km and 30 km. The accompanying map shows two sets of the anomalies for $T = 30$ km viz. one for local compensation and the second for a spreading of the compensation over an area up to a radius $R$ of 116.2 km; the last set has been underlined. In the near future the writer will give a more detailed investigation of our area in the report about all the gravity results obtained at sea to be published by the Netherlands Geodetic Commission; this will contain more anomaly maps as well as gravimetric profiles. Here we shall only give a short summary of this investigation.

A careful study of the different sets of anomalies shows that part of the topographic features seems to be more or less locally compensated and another part regionally. For Madeira the former investigation already mentioned has shown that the anomalies point

[1]) Dr. FR. V. WOLFF, Der Vulkanismus, Bd. II, pp. 959 and 971, Stuttgart, 1931.

[2]) For Madeira see the first foot-note on the preceding page.

[3]) F. A. VENING MEINESZ, Tables for regional and local isostatic reduction (Airy system), Publ. Neth. Geod. Comm. Waltman (Mulder), Delft, 1941.

to a large degree of regionality. Adopting the island to consist of heavy volcanic material of a density $\theta$ of 2.937 we obtain a value of $R$ of 232.4 km and putting $\theta$ at 3.07 we find 174.3 km. A similar result has been found for Hawaii, Oahu, Bermudas, São Vicente (Cape Verde Is), Canary Is and Mauritius. In view of these results on volcanic islands it is remarkable that those islands of the Azores where gravity has been observed, i.e. São Miguel and Fayal, show a much smaller degree of regionality. Adopting the same values for the density $\theta$ we find the anomalies on São Miguel to get into harmony with those in adjacent waters for a value of $R$ of less than 100 km, while for Fayal the results give no clear indication but probably they point to a still smaller degree of regionality. It appears to the writer that we can well understand this difference from the situation for other volcanic islands; we could explain it by the many fault-planes in the Azores reducing the coherence of the crust.

The same uncertainty as found for Fayal is experienced when studying the results for the submarine banks west of the Iberian peninsula. As the map shows gravity profiles have been made for the Josephine Bank and for the submarine promontory west of Cape Finisterre. Over the first bank the profile to the west seems to indicate regional compensation and that to the north local compensation. Over the second the southern profile appears to show local and the northern one regional compensation. Here also we probably may attribute this irregular result to faulting which in some directions diminishes the crust's resistance to local adjustment.

During one of the expeditions pairs of stations have been observed at small distances of about 10—20 km from each other. This occurred during the winter voyage of Hr. Ms. O 16 which took place in unusual bad weather. This induced Captain Van Waning to give his crew from time to time a few hours of rest and a quiet meal by staying submerged during a longer time than usual. The writer availed himself of this opportunity to repeat the observations at the end of this time. We find two of these pairs to the NW and to the W of Cape Finisterre, one near Terceira, one to the SSW of Fayal, one near Flores and one to the SW of the Azores at about 34½° N.L. It is interesting to see that nearly all these pairs show considerable differences of depth and so they make it possible to study the way of isostatic compensation of these irregularities in the topography although the mean error of 5—8 milligals does not allow strong conclusions. Still the map shows that nearly all of them point to regional compensation. Only the pairs to the SSW of Fayal and to the SW of the Azores do not allow a conclusion as the difference of the two anomalies do not vary much for the regional and the local reduction. The final report will contain the detailed results for these pairs of stations and will give the anomalies for all the reductions. The fact that these irregularities in the topography appear to be regionally compensated seems to point to their not having been brought about by faulting along vertical fault-planes which would probably lead to more or less local compensation. They may have been caused by volcanic activity but also by faulting under lateral compression along tilted planes; both these origins may be expected to bring about regional compensation.

To the west of the Azores, i.e. to the west of the Mid Atlantic Rise and about parallel to it, a series of stations over deep water shows positive anomalies; the sea-depth ranges here from 4000 to 5000 meters. The amount of these anomalies is smallest for the local reduction and the WE gravimetric profile across it gives the same result; for the regional reduction, especially for large values of $R$, this profile shows a bulge of larger positive anomalies to the west of the Mid Atlantic Rise and this disappears for the local reduction. For $T = 20$ km this is still more the case than for $T = 30$ km. So this seems to indicate that the Mid Atlantic Rise is locally compensated corresponding to small values of $T$.

As the map shows, a similar result follows from the profile to the SW of the Azores at about 34° N.L.; the three stations to the left, over the western slope of the Mid Atlantic Rise, show better agreement for local than for regional reduction. Other crossings at latitudes of 23° N.L. and 10° N.L. give the same result although the great distance of the stations and the uncertainty of the submarine topography does not allow a strong

conclusion. Still the mutual agreement of all the profiles gives us some confidence that our result is justified.

About its meaning we may make more than one supposition. In the first place it may mean that this western slope of the Mid Atlantic Rise has the same character as the slopes of the edges of the continents. In a previous paper [1]) the writer has mentioned the gravity results observed over these slopes. Nearly all of them give the same result; they show these slopes to be locally compensated according to small values of the crustal thickness $T$ of 20—30 km. The most obvious explanation, in harmony also with the seismic results, is to suppose the granite layer of the crust to become suddenly much thinner at the edge of the continent or even to disappear there. As it is generally admitted that in the central and eastern Atlantic a granite layer is present in the crust, it might be possible that the explanation also applies to the western and eventually also to the eastern slope of the Mid Atlantic Rise. The small values found for $T$ rather point in this direction. We may, however, also suppose a relative movement in vertical sense along a vertical fault-plane of the Rise with regard to the area west of it brought about by some change of density, a sinking of the last part or a rising of the first. This would likewise involve a local isostatic compensation of the slope.

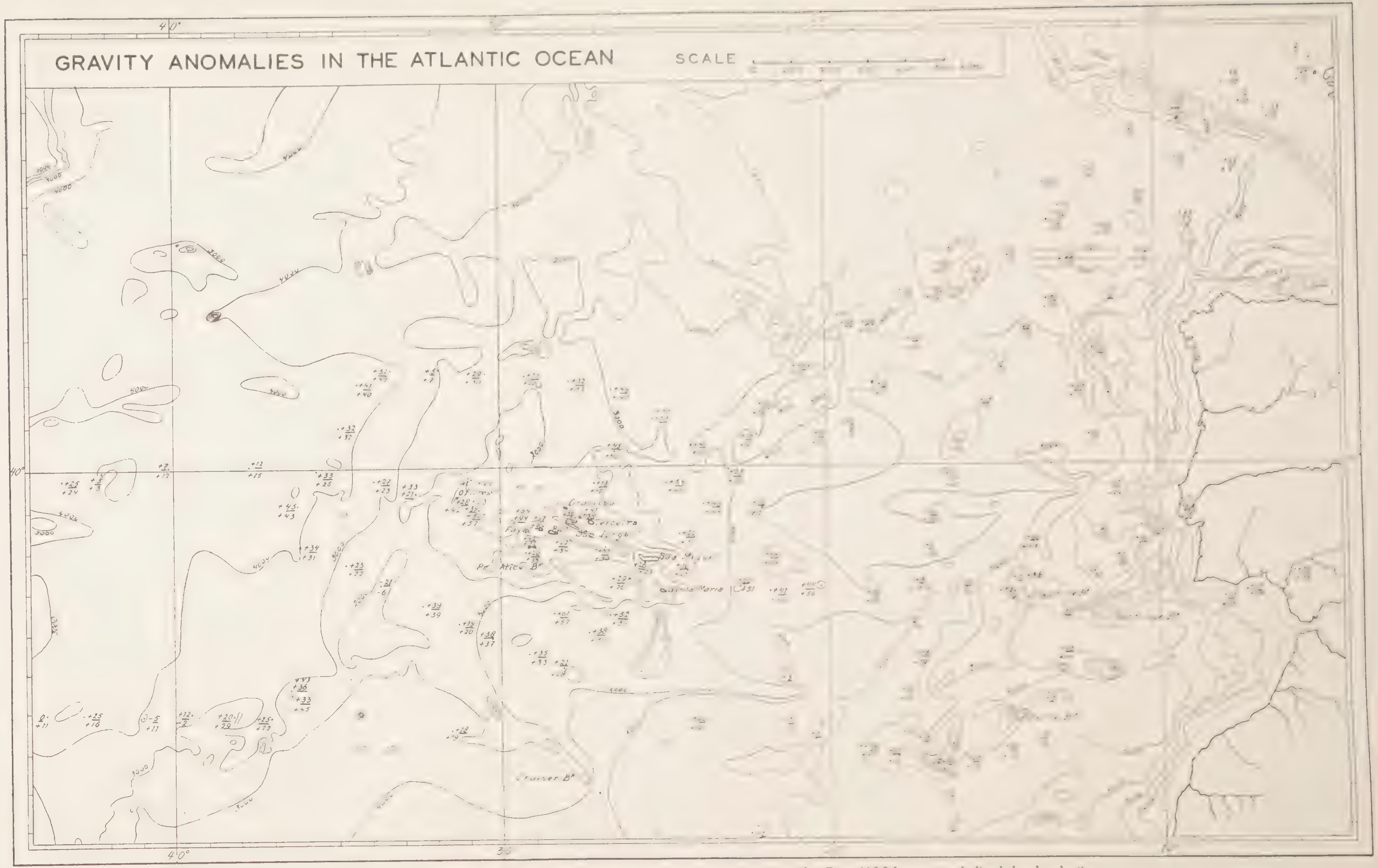
These are the main results obtained by the comparison of the anomalies according to local and regional reduction. We shall now consider other features of the anomaly fields.

In the first place we have already mentioned that notwithstanding the seismicity of our area no trace is found of belts of strong negative anomalies similar to those found in the East and West Indies and near Japan. This is a strong indication that the tectonic phenomena in our area have a different character and that there is no question here of a downward buckling of the Earth's crust nor probably of folding and overthrusting of the surface layers.

In the second place an important feature of the anomaly-field is the presence of fields of positive anomalies all showing a striking correlation with the topography; they more or less coincide with elevated parts of the ocean-floor. This is e.g. clearly shown by the area of the Azores where the line of + 30 milligal in the map of the locally reduced anomalies can be compared to the contour-line of 3000—4000 m depth. They do not exactly coincide and in some cases the anomalies extend somewhat further than the elevation, e.g. to the NE of São Miguel, but the correlation can not be doubted. Other examples are given by the ridge to the NE of the Azores, by that of the Gorringe Bank and the Josephine Bank, by the bank to the W of Cape Finisterre and by the short E—W ridge of Madeira. This does not mean a close correlation of the elevation and the anomalies; the higher elevations, e.g. the islands of the Azores and Madeira and the high banks, do not show corresponding high positive anomalies, even not in the field corresponding to local compensation. The correlation seems more to have regard to the regional elevation than to the local topographic features. Its being more or less independent of the type of reduction, local or regional, points in the same direction.

It is difficult to find an explanation of this disturbance of the isostatic equilibrium as well as of the remarkable correlation to the regional topography which seems too clear to be fortuitous. In connection with the volcanic processes at the surface we might suppose these anomalies to be caused by the rising of heavy magmatic material in the deeper layers bringing about a rising of the whole area without the isostatic adjustment keeping pace with it. This would imply a fairly great speed of the phenomenon but this of course would apply to other explanations as well. The mean anomaly in the area of the Azores of + 45 milligal corresponds to an uncompensated rock-layer of 580 meters of a density of 2.67 (to be diminished for the computation by the density of sea-water of 1.028). Applying to these data the formula for the readjustment of isostatic equilibrium

[1]) F. A. VENING MEINESZ, Gravity over the continental edges, Proc. Ned. Akad. v. Wetensch. Amsterdam, **44**, 8 (1941).




Isostatic anomalies for a thickness of the crust $T$ of 30 km, underlined: regional reduction for $R = 116.2$ km, not underlined: local reduction.

derived by the writer from the post-glacial uplift of Scandinavia [1]) and introducing in these formulas a diameter $L$ of 400 km, we find the sinking we might expect if no other phenomena were taking place to be about one cm pro year. This may give an idea of the speed of the phenomenon needed for counterbalancing this sinking and keeping up the positive anomalies in their present size. The geological evidence on the islands of the Azores does not appear to point to a sinking but rather to a rising and these movements seem to be irregular and of a smaller amount than the above figure. Generally speaking the eastern parts of the islands seem to have risen more than the western parts and the eastern islands more than the western.

The tentative explanation mentioned above goes more or less in the same direction as the hypothesis given by CLOOS [2]) in an investigation mainly based on the geometrical pattern of the topography in the Azores, but with this difference that CLOOS looks upon the rising magmatic bulge in the archipelago as the cause of the faulting and of the volcanism because of the tension it brings about in the crust, while the writer, considering the two directions apparent in the submarine topography of the whole area from the Azores to Europe as mentioned in the beginning of this paper, is inclined to think the faulting to be the primary cause and the rising of the magma together with the volcanism to be brought about by the presence of these fault-planes.

For the further investigation of these interesting problems it is important to make more soundings and a more detailed gravimetric survey of these areas of positive anomalies in order to get a better idea of the character and extension of these fields and of their correlation to the topography.

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[1]) F. A. VENING MEINESZ, The determination of the Earth's plasticity from the post-glacial uplift of Scandinavia, Isostatic adjustment, Proc. Ned. Ak. v. Wet. Amsterdam, 40, 8, p. 662 (1937).

[2]) H. CLOOS, Zur Tektonik der Azoren, Abh. Preuss. Akad. d. Wiss. Phys. Math. Kl. 1939, 5 (see foot-note first page).

**Hydrodynamics.** — *On the influence of the concentration of a suspension upon the sedimentation velocity* (in particular for a suspension of spherical particles) *). By J. M. BURGERS. (Mededeeling No. 42 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hoogeschool te Delft.)

(Communicated at the meeting of January 31, 1942.)

**23.** In the preceding part of this paper it was mentioned that the case of a suspension enclosed in a vessel requires a separate investigation. It is found that there is one case only in which the problem can be treated in a simple way; this is the case when the suspension is enclosed between two parallel plane walls, both being perpendicular to the $x$-axis.

It is well known that the general problem of the influence of the walls of a vessel upon the motion of a single particle in a viscous liquid constitutes a difficult subject, which has been investigated by many authors [21]). It is found that correction terms must be added to STOKES' resistance formula, which terms in general are of the order $a/l$, $a$ being the radius of the particle and $l$ the distance of the particle from the nearest wall. In the present investigation, however, we are not concerned with effects of this order of magnitude, *i.e.* with effects which depend upon the ratio of the dimensions of the vessel to the diameter of a particle: our object is to determine the effects which are proportional to the number of particles per unit volume in the suspension, and it is asked whether these effects may suffer some influence from the circumstance that the suspension is enclosed between fixed walls.

The presence of the vessel introduces the boundary condition that all three components of the velocity of the liquid must be zero at the walls. Besides the wall has an influence upon the spatial distribution of the particles: even if there are no repulsive forces between the wall and the particles, no particle can have its centre within a layer of thickness $a$ along the wall.

The first point that asks for investigation concerns the influence which the boundary conditions may have upon the field of flow considered in sections **10.—18.** It has been proved in **9.**, in deducing the equations for the flow connected with a single particle, that the integral $\iint dy\, dz\, u$, extended over an infinite plane $x = constant$, has the value zero. From the symmetry of the field it follows that the integrals $\iint dy\, dz\, v$ and $\iint dy\, dz\, w$ over a plane $x = constant$ likewise will be zero. Consequently, when a suspension is enclosed between two plane walls, both of which are perpendicular to the $x$-axis, the field of flow calculated by means of the formulae developed in **9.** will already satisfy the boundary conditions in an average way. When a more exact solution should

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*) Continued from these Proceedings **45**, 1942, p. 16. — It should be mentioned that the results referring to a cloud of particles, obtained in sections **19.—22.**, are similar to those given by M. S. VON SMOLUCHOWSKI, Proc. Vth Intern. Congr. of Mathem. (Cambridge 1912), Vol. II, p. 192.

[21]) The motion of a sphere in the neighbourhood of a plane wall has been treated by H. A. LORENTZ, Abhandlungen über theoretische Physik I (Leipzig 1907), p. 40. For further references see: H. LAMB, Hydrodynamics (6th Ed.) Cambridge 1932, p. 598, footnote †; H. FAXÉN, Einwirkung der Gefässwände auf den Widerstand gegen die Bewegung einer kleinen Kugel in einer zähen Flüssigkeit, Dissertation Upsala 1921; C. W. OSEEN, Hydrodynamik (Leipzig 1927), pp. 140, 144, 190, 196.

be aimed at, it would be necessary to introduce local corrections only, which presumably will not have an observable influence at distances from the walls greater than a few times the average distance between neighbouring particles. Hence it is probable that in this case the presence of the walls will have no particular influence upon the concentration effect.

Moreover, the fact that the field is bounded by planes perpendicular to the $x$-axis affords an extra justification for calculating the value of the integral $n\iiint dx\,dy\,dz\,u$, which occurred in section **17.**, by integrating first with respect to $dy$ and $dz$ and afterwards with respect to $dx$. We may conclude, therefore, that the result obtained in section **18.** will apply to the present case.

It is of interest to observe that the same conclusion is arrived at when we return to the method originally developed in sections **5.**—**7.** From the equation of continuity it follows that along a wall perpendicular to the $x$-axis the condition $\partial u/\partial n = 0$ is fulfilled simultaneously with the condition $u = 0$. Hence in eq. (22) the integral $\iint dS_e\, \partial\Phi/\partial n$, the value of which had been denoted by $k_3$, disappears. At the same time it follows that within a layer of thickness $a$ along the wall the function $\Phi$ can be at most of the order $a^2 r^{-3}$, $r$ being the distance of a point of the wall from the centre of the particle. Consequently the integral $\iiint_{G^*} dx\,dy\,dz\;\Phi_c$ in formula (20), which integral was extended over such a layer and which had been represented by $-k_2 r_0^2$, can be neglected [22]. Hence in the case of a suspension between two plane walls, both of which are perpendicular to the $x$-axis, the coefficients $k_2$ and $k_3$ in eq. (24) both become zero. As has been mentioned in footnote 15) to section **17.** equation (24) then leads to the result (62$a$).

**24.** When the suspension is enclosed in a vessel of arbitrary form a similar treatment cannot be given. In section **7.** it has been found that when we start from STOKES's formulae for the flow produced by a moving sphere, a definite result can be obtained only when we should know the value of the derivative $\partial\Phi/\partial n$ along the walls of the vessel. It may be possible to determine this function in a few cases (although with much labour), but a convenient expression, enabling the evaluation of the integrals in a general way, apparently does not exist. It is probable that the value of the expression (24) remains of

22) Making use of the formulae developed by LORENTZ (*l.c.* footnote 21 above), it is found that for a spherical particle in the neighbourhood of a single wall perpendicular to the $x$-axis the function $\Phi$ assumes the form:

$$\Phi = \frac{1}{r_1} + \frac{(l-\nu)^2}{r_1^{\,3}} - \frac{1}{r_2} - \frac{l^2+\nu^2}{r_2^{\,3}} - \frac{6\,l\,\nu\,(l+\nu)^2}{r_2^{\,5}},$$

terms of the order $a^2/r^3$ being neglected. Here $l$ is the distance of the centre of the sphere from the wall; $\nu$ is the distance of the point considered from the wall; $r_1$ is the distance from the centre of the sphere to this point; $r_2$ is the distance from the image of the centre in the wall to this point. When $\nu$ is sufficiently small the expression can be developed and becomes:

$$\Phi = \nu^2\,(-18\,l^2/r^5 + 30\,l^4/r^7) + \ldots$$

It is found that:

$$\iint dy\,dz\,(-18\,l^2/r^5 + 30\,l^4/r^7) = 0,$$

when the integral is extended over the infinite wall.

the order $a^2$; in that case the order of magnitude of the correction to be applied to the sedimentation velocity remains the same as that given in equation (64), but the value of the coefficient $\lambda_{II}$ is not known.

It is probable that the value of the integral $\iint dS_e \, \partial\Phi/\partial n$, occurring in (22), is connected with the resultant of the frictional forces acting upon the walls of the vessel. In general the weight of the suspension will be carried partly by pressures, partly by frictional forces acting upon the walls. When the whole weight is carried by the resultant of the pressures (this is the case for a suspension enclosed between two parallel plane walls, perpendicular to the $x$-axis, as in the case considered in section **23.**), it is possible that formula (64) will apply with the value of $\lambda_{II}$ as given in **17.** When the weight is partly or wholly carried by the frictional forces, the result perhaps may be different, and in this way an influence of the shape of the vessel could be experienced.

The application of a point of view, related to that of section **10.**, does not appear to be more promising. We might decompose the system of forces acting upon the liquid and the particles into the following components:

*a*) a continuous field of force having the intensity $\varrho g + nF$ per unit volume, acting through the whole space, and balanced by a pressure gradient of magnitude $\partial p/\partial x = \varrho g + nF$;

*b*) a set of "equilibrium systems" of the type considered in **9.**, each system having its centre at the centre of a particle;

*c*) a continuous field of force acting in a thin layer along the walls, making up for the "diffuse fields" of those "equilibrium systems" which would influence the field inside the vessel, if the suspension was imagined to extend also through and beyond the walls. It should be assumed again that the parameter $\varkappa$ is chosen in such a way, that $1/\varkappa$, while being large in comparison with the average distance between neighbouring particles, at the same time will be small in comparison with the dimensions of the vessel and with the radius of curvature of the walls.

In attempting to work out the equations for the motion of the liquid upon this basis, there again occur difficulties with integrals of the type $\iint dS_e \, u$ ($dS_e$ being an element of the wall).

The difficulties probably will increase, when the number of particles per unit volume in the immediate neighbourhood of the wall should be different from the number in the more interior part of the vessel.

Provisionally the problem must be left here, in the hope that a more efficient method may be found at some later time.

**Mathematics.** — *A remarkable family.* By J. G. VAN DER CORPUT.

(Communicated at the meeting of January 31, 1942.)

## CHAPTER III.

### *On analytical solutions of functional systems* [1]).

Let us consider a functional system of the form

$$g_\varrho \{x, f_\nu (l_\varkappa (x))\} = 0.$$

In this chapter $\varkappa$ runs through the values 1, 2, ..., $k$, where $k$ denotes an integer $\geqq 2$, while $\nu$, $\varrho$ and $\sigma$ run through the values 1, 2, ..., $n$, where $n$ is a positive integer. The functional system involves $k$ given functions $l_\varkappa (x)$ of a variable $x$, in addition $n$ given functions $g_\varrho (x, y_{\varkappa\nu})$ of the $1 + kn$ variables $x$, $y_{\varkappa\nu}$ and finally $n$ unknown functions $f_\nu (x)$ of $x$. I say that the functional system possesses a solution $(f_\nu (x))$, analytical and vanishing at the origin $x = 0$, if the $n$ funtions $f_\nu (x)$ are analytical at the origin, take at that point the value zero and satisfy the considered functional system in the vicinity of the origin.

The following examples show that several different cases are possible.

*A.* Dealing with the functional equation

$$f_1 (x) = f_1 \left(\frac{x}{2}\right) + \log \frac{1-x}{1-\frac{x}{2}},$$

we have

$$n = 1,\ k = 2,\ l_1 (x) = \frac{x}{2},\ l_2 (x) = x$$

and

$$g_1 (x, y_{11}, y_{21}) = y_{21} - y_{11} - \log \frac{1-x}{1-\frac{x}{2}}.$$

Trying

$$f_1 (x) = \sum_{\alpha=1}^{\infty} F(\alpha)\, x^\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

we obtain

$$F(\alpha) \left(1 - \frac{1}{2^\alpha}\right) = \frac{-1}{\alpha} \left(1 - \frac{1}{2^\alpha}\right) \quad (\alpha \geqq 1),$$

hence

$$F(\alpha) = \frac{-1}{\alpha} \quad \text{and} \quad f_1 (x) = \log (1-x).$$

[1]) Chapter I and the first part of chapter II have been published in Euclides **18** (1941—42), p. 50—78; the rest of chapter II is about to appear in the same periodical. For the well understanding of this paper it is not necessary that the reader is acquainted with the chapters I and II. The remarkable family consists of the functions characterised by functional equations.

The functional equation possesses one and only one solution analytical and vanishing at the origin.

*B.* Considering the functional equation

$$f_1(x+x^2)=f_1(x)+f_1\left(\frac{x}{2}\right)-\frac{x}{2},$$

we obtain

$$n=1,\ k=3,\ l_1(x)=x+x^2,\ l_2(x)=x,\ l_3(x)=\frac{x}{2}$$

and

$$g_1(x, y_{11}, y_{21}, y_{31})=y_{11}-y_{21}-y_{31}+\frac{x}{2}.$$

Trying again (1) we find for $\alpha \geqq 2$

$$F(\alpha)=2^{\alpha}\sum_{\frac{\alpha}{2}\leqq\beta<\alpha}\binom{\beta}{2\beta-\alpha}F(\beta),$$

hence

$$F(\alpha)\geqq 2^{\alpha}(\alpha-1)F(\alpha-1).$$

From $F(1)=1$ it follows that

$$F(\alpha)\geqq 2^{\frac{1}{2}(\alpha-1)(\alpha+2)}(\alpha-1)!\quad(\alpha\geqq 1).$$

The coefficients $F(\alpha)$ are defined unambiguously, but the radius of convergence of the found power series equals zero. The functional equation does not possess any solution analytical and vanishing at the origin.

*C.* Dealing with $f_1(x)-f_1(-x)=x^2$ and trying (1), we find relations involving the coefficients $F(\alpha)$, but these relations are contradictory. Indeed, consideration of the coefficient of $x^2$ gives $0=1$. It is even clear that the functional equation does not possess any solution in the vicinity of the origin, the left-hand side being an odd, the right-hand side an even function of $x$.

*D.* The functional equation $f(x)=-f(-x)$ possesses an infinity of solutions, analytical and vanishing at the origin, for any odd function is a solution.

Let us now return to the original functional system. By $\left(\frac{\delta g_{\varrho}}{\delta y_{\varkappa\nu}}\right)_0$ I denote the value of that partial derivative at the origin $x=y_{\varkappa\nu}=0$ of the $(1+kn)$ — dimensional espace; $\triangle$ denotes the determinant of $n$ rows and columns, in which the constituent in the $\varrho^{\text{th}}$ row and $\nu^{\text{th}}$ column has the value $\left(\frac{\delta g_{\varrho}}{\delta y_{k\nu}}\right)_0$; finally $D_\alpha$ denotes the determinant of $n$ rows and columns, in which the constituent in the $\varrho^{\text{th}}$ row and $\nu^{\text{th}}$ column is

$$\sum_{\varkappa}\left(\frac{\partial g_{\varrho}}{\partial y_{\varkappa\nu}}\right)_0 (l'_{\varkappa}(0))^{\alpha}.$$

**Theorem I.** *Conditions.* (1) *Suppose that the k given functions $l_\varkappa(x)$ are analytical at the origin $x=0$ and assume at that point the value zero, that $g_\varrho(x, y_{\varkappa\nu})$ is analytical and vanishes at the origin $x=y_{\varkappa\nu}=0$,* that $\triangle\neq 0$ and

$$|l'_{\mu}(0)|<|l'_k(0)|\quad(\mu=1, 2, \ldots, k-1).$$

(2) *Let us suppose in addition*

$$D_\alpha \neq 0 \quad (\alpha = 1, 2, \ldots).$$

*Then the functional system possesses one and only one solution analytical and vanishing at the origin.*

The proof runs as follows. The notation

$$p(x) << q(x)$$

signifies that

$$p(x) = P(0) + P(1)\,x + P(2)\,x^2 + \ldots$$

and

$$q(x) = Q(0) + Q(1)\,x + Q(2)\,x^2 + \ldots$$

are functions of $x$ analytical at the origin with the property

$$|P(\alpha)| \leqq Q(\alpha) \quad (\alpha = 0, 1, \ldots).$$

The notation

$$p(x, y_{\varkappa\nu}) << q(x, y_{\varkappa\nu})$$

signifies that

$$p(x, y_{\varkappa\nu}) = \sum_{\beta, \gamma_{\varkappa\nu}} P(\beta, \gamma_{\varkappa\nu})\, x^\beta \prod_{\varkappa, \nu} y_{\varkappa\nu}^{\gamma_{\varkappa\nu}}$$

and

$$q(x, y_{\varkappa\nu}) = \sum_{\beta, \gamma_{\varkappa\nu}} Q(\beta, \gamma_{\varkappa\nu})\, x^\beta \prod_{\varkappa, \nu} y_{\varkappa\nu}^{\gamma_{\varkappa\nu}}$$

are functions of the $1 + kn$ variables $x$, $y_{\varkappa\nu}$ analytical at the origin $x = y_{\varkappa\nu} = 0$ with the property

$$|P(\beta, \gamma_{\varkappa\nu})| \leqq Q(\beta, \gamma_{\varkappa\nu});$$

$\beta$ and $\gamma_{\varkappa\nu}$ ($\varkappa = 1, \ldots, k$; $\nu = 1, \ldots, n$) run through the sequence of the integers $\geqq 0$.

By hypothesis there exists a $(1 + kn)$ — dimensional vicinity of the origin $x = y_{\varkappa\nu} = 0$ with the property that the $n$ functions $g_\varrho(x, y_{\varkappa\nu})$ are defined and analytical at every point $(x, y_{\varkappa\nu})$ of $V$. The determinant $\triangle$ being $\neq 0$, there exists a $(1 + (k-1)\,n)$-dimensional vicinity $V_1$ of the point $x = y_{\mu\nu} = 0$ (in this chapter $\mu$ runs always through the values $1, 2, \ldots, k-1$) such that $n$ analytical functions $h_\varrho(x, y_{\mu\nu})$ may be found in $V_1$ with the following properties; (1) if $(x, y_{\mu\nu})$ is an arbitrary point in $V_1$ and we put

$$y_{k\varrho} = h_\varrho(x, y_{\mu\nu}),$$

then the point $(x, y_{\varkappa\nu})$ lies in $V$ and satisfies the $n$ relations

$$g_\varrho(x, y_{\varkappa\nu}) = 0.$$

(2) The $n$ functions $h_\varrho(x, y_{\mu\nu})$ assume at $x = y_{\mu\nu} = 0$ the value zero.

(3) The $n$ analytical functions $h_\varrho(x, y_{\mu\nu})$ are defined unambiguously in $V_1$ by the properties (1) and (2).

The given functional system is in the vicinity of the origin $x = 0$ equivalent to

$$f_\varrho(l_k(x)) = h_\varrho\{x, f_\nu(l_\mu(x))\},$$

in other words: any system of $n$ analytical functions $f_\nu(x)$ vanishing at the origin and satisfying one of both functional systems, satisfies also the other.

From condition (1) it follows that $l'_k(0) \neq 0$, so that the substitution $l_k(x) = t$ gives in the vicinity of the origin an analytical (1, 1) transformation. Hence $x = q(t)$ and $l_\mu(x) = w_\mu(t)$ are analytical functions of $t$ at $t = 0$ and the functional system reduces to

$$f_\varrho(t) = h_\varrho\{q(t), f_\nu(w_\mu(t))\}.$$

We can write

$$w_\mu(t) = \sum_\beta W_\mu(\beta)\, t^\beta \quad \text{and} \quad q(t) = \sum_\gamma Q(\gamma)\, t^\gamma,$$

where $\beta$ and $\gamma$ run through the sequence of the positive integers, and

$$h_\varrho(x, y_{\mu\nu}) = \sum_{\delta, \zeta_{\mu\nu}} H_\varrho(\delta, \zeta_{\mu\nu})\, x^\delta \prod_{\mu,\nu} y_{\mu\nu}^{\zeta_{\mu\nu}},$$

where $\delta$ and $\zeta_{\mu\nu}$ run through the sequence of the integers $\geqq 0$.

First, assume that the functional system possesses a solution

$$f_\nu(t) = \sum_\alpha F_\nu(\alpha)\, t^\alpha,$$

analytical and vanishing at the origin; $\alpha$ runs through the sequence of positive integers. Then we have in the vicinity of the origin

$$\sum_\eta F_\varrho(\eta)\, t^\eta = \sum_{\delta, \zeta_{\mu\nu}} H_\varrho(\delta, \zeta_{\mu\nu}) \left(\sum_\gamma Q(\gamma)\, t^\gamma\right)^\delta X(\zeta_{\mu\nu}),$$

where

$$X(\zeta_{\mu\nu}) = \prod_{\mu,\nu} \left(\sum_\alpha F_\nu(\alpha) \left(\sum_\beta W_\mu(\beta)\, t^\beta\right)^\alpha\right\}^{\zeta_{\mu\nu}};$$

$\eta$ runs through the sequence of the positive integers. The expansion of the right-hand side in powers of $t$ produces the coefficient $F_\varrho(\eta)$ of $t^\eta$ written as a sums of terms. To find one of these terms I consider a certain $\mu$ $(1 \leqq \mu \leqq k-1)$ and a certain $v$ $(1 \leqq v \leqq n)$ and I take $\delta = 0$, $\beta = 1$, $\alpha = \eta$, $\zeta_{\mu\nu} = 1$, the other exponents $\zeta = 0$. In the term found in this manner we have

$$W_\mu(\beta) = w'_\mu(0) \quad \text{and} \quad H_\varrho(\delta, \zeta_{\mu\nu}) = \left(\frac{\partial h_\varrho}{\partial y_{\mu\nu}}\right)_0$$

and the term in question is therefore

$$\left(\frac{\partial h_\varrho}{\partial y_{\mu\nu}}\right)_0 F_\nu(\eta)\, (w'_\mu(0))^\eta.$$

In this manner we find that $F_\varrho(\eta)$ is equal to

$$\sum_{\mu,\nu} \left(\frac{\partial h_\varrho}{\partial y_{\mu\nu}}\right)_0 F_\nu(\eta)\, (w'_\mu(0))^\eta$$

augmented by the sum of the other terms; this sum is a polynomial $u_\varrho(F_\sigma(\alpha))$ in the numbers $F_\sigma(\alpha)$, where $\sigma$ runs through $1, 2, \ldots, n$ and $\alpha$ runs through $1, 2, \ldots, \eta - 1$. Thus we obtain

$$F_\varrho(\eta) = \sum_\nu F_\nu(\eta) \sum_\mu \left(\frac{\partial h_\varrho}{\partial y_{\mu\nu}}\right)_0 (w'_\mu(0))^\eta + u_\varrho(F_\sigma(\alpha)). \quad . \quad . \quad . \quad (2)$$

If the coefficients $F_\sigma(\alpha)$ $(\alpha < \eta)$ are already known, we find $n$ linear equations with $n$

unknown coefficients $F_\nu(\eta)$. The determinant $E(\eta)$ of this system of equations possesses $n$ rows and columns; the constituent in the $\varrho^{\text{th}}$ row and $\nu^{\text{th}}$ column is

$$\varepsilon_{\varrho\nu} - \sum_\mu \left(\frac{\partial h_\varrho}{\partial y_{\mu\nu}}\right)_0 (w'_\mu(0))^\eta,$$

where

$$\begin{aligned} \varepsilon_{\varrho\nu} &= 1 \quad \text{for} \quad \varrho = \nu \\ &= 0 \quad \text{for} \quad \varrho \neq \nu. \end{aligned}$$

Using

$$w'_\mu(0) = \frac{l'_\mu(0)}{l'_k(0)},$$

we observe that the constituent in the $\varrho^{\text{th}}$ row and $\nu^{\text{th}}$ column of the determinant $(l'_k(0))^{n\eta} E(\eta)$ has the value

$$(l'_k(0))^\eta \varepsilon_{\varrho\nu} - \sum_\mu \left(\frac{\partial h_\varrho}{\partial y_{\mu\nu}}\right)_0 (l'_\mu(0))^\eta.$$

Since $y_{k\varrho} = h_\varrho(x, y_{\mu\nu})$ satisfies the system $g_\sigma(x, y_{\varkappa\nu}) = 0$, we have

$$\left(\frac{\partial g_\sigma}{\partial y_{\mu\nu}}\right)_0 + \sum_\varrho \left(\frac{\partial g_\sigma}{\partial y_{k\varrho}}\right)_0 \left(\frac{\partial h_\varrho}{\partial y_{\mu\nu}}\right)_0 = 0.$$

$\triangle$ being a determinant of $n$ rows and columns, in which the constituent in the $\sigma^{\text{th}}$ row and $\varrho^{\text{th}}$ column equals $\left(\dfrac{\delta g_\sigma}{\delta y_{k\varrho}}\right)_0$, the product of $(l'_k(0))^{n\eta} E(\eta)$ and $\triangle$ is the determinant, whose constituent in the $\sigma^{\text{th}}$ row and $\nu^{\text{th}}$ column has the value

$$\begin{aligned} &(l'_k(0))^\eta \sum_\varrho \left(\frac{\partial g_\sigma}{\partial y_{k\varrho}}\right)_0 \varepsilon_{\varrho\nu} - \sum_\mu (l'_\mu(0))^\eta \sum_\varrho \left(\frac{\partial g_\sigma}{\partial y_{k\varrho}}\right)_0 \left(\frac{\partial h}{\partial y_{\mu\nu}}\right)_0 \\ &= \left(\frac{\partial g_\sigma}{\partial y_{k\nu}}\right)_0 (l'_k(0))^\eta + \sum_\mu \left(\frac{\partial g_\sigma}{\partial y_{\mu\nu}}\right)_0 (l'_\mu(0))^\eta = \sum_\varkappa \left(\frac{\partial g_\sigma}{\partial y_{\varkappa\nu}}\right)_0 (l'_\varkappa(0))^\eta, \end{aligned}$$

hence

$$D(\eta) = E(\eta)\,(l'_k(0))^{n\eta} \triangle \qquad (\eta = 1, 2, \ldots). \quad . \quad . \quad . \quad . \quad (3)$$

$D(\eta)$ being $\neq 0$, we find $E(\eta) \neq 0$. Hence: if the coefficients $F_\sigma(\alpha)$ $(\alpha < \eta)$ are already known, the $n$ coefficients $F_\nu(\eta)$ are defined unambiguously by the $n$ relations (2). In this manner we have proved:

*The functional system possesses at most one solution analytical and vanishing at the origin.*

By means of the recurrent relations (2) we can determine the coefficients $F_\nu(\eta)$. Thus we obtain $n$ formal power series $\sum_\alpha F_\nu(\alpha)\, t^\alpha$. To prove the theorem it is sufficient to show that each of these power series possesses a positive radius of convergence. In fact, these power series give then a solution, analytical and vanishing at the origin.

As we have shown, the determinants $E(\eta)$ $(\eta = 1, 2, \ldots)$ differ from zero. It follows from $|w'_\mu(0)| < 1$, that for $\eta \to \infty$ each constituent in the principal diagonal of $E(\eta)$ tends to 1, each other constituent to zero. Therefore $E(\eta)$ tends to 1 and $|E(\eta)|$

possesses a positive lower bound independent of $\eta$. Each constituent of $E(\eta)$ being bounded, we deduce from (2)

$$|F_\nu(\eta)| \leqq \frac{A}{n} \sum_{\varrho} |u_\varrho(F_\sigma(\alpha))|, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $A$ denotes an appropiate number independent of $\eta$,

Since $|w'_\mu(0)| < 1$, we have

$$|w'_\mu(0)| \leqq \Theta < 1,$$

where $\Theta$ is a conveniently chosen positive number $< 1$. Since $w_\mu(x)$ and $q(x)$ are analytical at $x = 0$, and $h_\varrho(x, y_{\mu\nu})$ at $x = y_{\mu\nu} = 0$, we have for sufficiently large $K$ and $B$

$$w_\mu(x) << \frac{\Theta x}{1 - Kx}, \qquad q(x) << \frac{Bx}{1 - Kx}$$

and

$$h_\varrho(x, y_{\mu\nu}) << \frac{B}{(1 - Kx) \prod_{\mu,\nu} (1 - K y_{\mu\nu})}.$$

I choose the positive number $\Gamma$ so small and then the positive number $\Lambda$ so large that

$$(1 + 2\Gamma)\left(\Theta + \frac{K}{\Lambda}\right) < 1 \quad \text{and} \quad \Gamma(K + \Lambda\Theta) \geqq KB \quad . \quad . \quad . \quad (5)$$

For each $z$ with absolute value $< \frac{1}{1+\Gamma}$ the function

$$(1 - z)^{1+(k-1)n} \{1 - (1 + \Gamma) z\}^{-1-(k-1)n}$$

possesses an expansion in powers of $z$. This expansion being valid for $z = \frac{1}{1+2\Gamma}$, we obtain for sufficiently large $M$

$$(1 - z)^{1+(k-1)n} \{1 - (1 + \Gamma) z\}^{-1-(k-1)n} << \frac{M}{1 - (1 + 2\Gamma) z}. \quad . \quad (6)$$

From (5) it appears that

$$\Lambda B M K (1 + 2\Gamma)^{N+1} \left(\Theta + \frac{K}{\Lambda}\right)^{N+1} < \Gamma \quad . \quad . \quad . \quad . \quad (7)$$

for sufficiently large positive integer $N$. Finally I choose $H \geqq \Lambda$ so large that the inequality

$$|F_\nu(\eta)| \leqq \frac{\Gamma}{K} H^\eta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

holds for $\nu = 1, 2, \ldots, n$ and $\eta = 1, 2, \ldots, N$. It is sufficient to show that this inequality holds for $\nu = 1, 2, \ldots, n$ and every positive integer $\eta$. In fact, then the $n$ power series $\sum_\alpha F_\nu(\alpha) t^\alpha$ possess a radius of convergence $\geqq \frac{1}{H}$. I may assume $\eta \geqq N + 1$ and suppose that the inequalities

$$|F_\nu(\alpha)| \leqq \frac{\Gamma}{K} H^\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

are proved already for any $\alpha < \eta$.

The following argument is based on inequality (4): $u_{\varrho}(F_{\varsigma}(a))$ is the coefficient of $t^{\eta}$ in the expansion of $h_{\varrho}\{q(t), j_{\nu}(w_{\mu}(t))\}$ where

$$j_{\nu}(x) = \sum_{\alpha < r_{\nu}} F_{\nu}(a)\, x^{\alpha}.$$

From (9) it follows that

$$j_{\nu}(x) << \frac{\Gamma H x}{K(1 - H x)},$$

hence

$$K\, j_{\nu}(w_{\mu}(t)) << \frac{\Gamma H \dfrac{\Theta t}{1 - K t}}{1 - H \dfrac{\Theta t}{1 - K t}} = \frac{\Gamma H \Theta t}{1 - (K + H \Theta) t} << \frac{\Gamma z}{1 - z},$$

where $z = (K + H\Theta)\, t$. Also from (5)

$$\Gamma (K + H \Theta) \geqq \Gamma (K + \Lambda \Theta) \geqq K B,$$

hence

$$K\, q(t) << \frac{K B t}{1 - K t} << \frac{\Gamma z}{1 - z}.$$

In this manner we obtain

$$h_{\varrho}\{q(t), j_{\nu}(w_{\mu}(t))\} << \frac{B}{\left(1 - \dfrac{\Gamma z}{1 - z}\right)^{1 + (k-1) n}}$$

$$= B (1 - z)^{1 + (k-1) n} \{1 - (1 + \Gamma) z\}^{-1 - (k-1) n} << \frac{B M}{1 - (1 + 2\Gamma) z}$$

by (6)

$$= \frac{B M}{1 - (1 + 2\Gamma)(K + H \Theta) t} << \frac{B M}{1 - (1 + 2\Gamma) H \left(\Theta - \dfrac{K}{\Lambda}\right) t}$$

$$<< B M \sum_{\alpha = 0}^{N} H^{\alpha} t^{\alpha} + \frac{\Gamma}{\Lambda K} \sum_{\alpha = N+1}^{\infty} H^{\alpha} t^{\alpha}$$

according to (5) and (7). The absolute value of the coefficient $u_{\varrho}(F_{\varsigma}(a))$ of $t^{\eta}$ in this expansion is therefore $\leqq \frac{\Gamma}{\Lambda K} H^{\eta}$ for every $\eta > N$, so that it follows from (4), that (8) holds for every $\eta > N$. This establishes the theorem.

*To be continued.*

**Mathematics.** — *On the uniqueness of solutions of differential equations.* By J. G. VAN DER CORPUT.

(Communicated at the meeting of January 31, 1942.)

**Theorem 1.** *Consider* $n+2$ *real numbers* $\xi, \eta_1, \ldots, \eta_{n+1}$, *further a positive number* $\omega$ *and the polynomial*

$$p(x)=\sum_{\nu=0}^{n} \eta_{\nu+1} \frac{(x-\xi)^{\nu}}{\nu!}.$$

*Let the real function* $f(x, y_1, \ldots, y_n)$ *be defined in the* $(n+1)$ — *dimensional strip*

$$\xi<x<\xi+\omega, -\infty<y_1<\infty, \ldots, -\infty<y_n<\infty$$

*in such a manner that any two points* $(x, y_1, \ldots, y_n)$ *and* $(x, Y_1, \ldots, Y_n)$ *of the region*

$$\xi<x<\xi+\omega; \quad \left|y_{\nu}-\frac{d^{\nu} p(x)}{d x^{\nu}}\right|<\omega \frac{(x-\xi)^{n-\nu}}{(n-\nu)!} \quad (\nu=1, \ldots, n)$$

*satisfy the inequality*

$$|f(x, Y_1, \ldots, Y_n)-f(x, y_1, \ldots, y_n)| \leqq \max_{\nu=1,\ldots,n} \frac{(n-\nu+1)!}{(x-\xi)^{n-\nu+1}}|Y_{\nu}-y_{\nu}|. \quad (1)$$

*Assertion*: If the positive number $\varepsilon$ is small enough, there exists in the interval $\xi \leqq x \leqq \xi+\varepsilon$ one function $y$ at most, which is $n$ times differentiable in that interval, which satisfies the differential equation

$$\frac{d^n y}{d x^n}=f\left(x, y, \frac{d y}{d x}, \ldots, \frac{d^{n-1} y}{d x^{n-1}}\right)$$

*in the interval* $\xi<x \leqq \xi+\varepsilon$ *and which satisfies the initial conditions*

$$y=\eta_1, \frac{d y}{d x}=\eta_2, \ldots, \frac{d^n y}{d x^n}=\eta_{n+1} \text{ for } x=\xi.$$

The special case $n=1$ is nearly equivalent to the theorem of Nagumo.

To prove the theorem I may assume that there exists in the interval $\xi \leqq x \leqq \xi+\alpha$, where $\alpha$ is an appropiate positive number $\leqq \omega$, one function $y=\varphi(x)$ at least with the mentioned properties.

Then

$$\frac{d^{n-1} \varphi(x)}{d x^{n-1}} \text{ and } \frac{d^n \varphi(x)}{d x^n}$$

take at $x=\xi$ the values $\eta_n$ and $\eta_{n+1}$, so that

$$\frac{1}{x-\xi}\left(\frac{d^{n-1} \varphi(x)}{d x^{n-1}}-\eta_n\right)$$

tends to $\eta_{n+1}$, if $x-\xi$ approaches zero. Hence we find in the interval $\xi<x \leqq \xi+\beta$, where $\beta$ is a convenient positive number $\leqq \alpha$,

$$\left|\frac{d^{n-1} \varphi(x)}{d x^{n-1}}-\eta_n-\eta_{n+1}(x-\xi)\right|<\omega(x-\xi),$$

i.e.

$$\left|\frac{d^{n-1}\varphi(x)}{dx^{n-1}}-\frac{d^{n-1}p(x)}{dx^{n-1}}\right|<\omega(x-\xi). \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

I shall show that any positive $\varepsilon \leq \beta$ possesses the required properties. Since the functions $\varphi(x)$ and $p(x)$ and their $1^{st}$, $2^{nd}$, ..., $(n-2)^{th}$ derivatives have at $x=\xi$ the same values, it follows from (2) by repeated integration that

$$\left|\frac{d^{\nu}\varphi(x)}{dx^{\nu}}-\frac{d^{\nu}p(x)}{dx^{\nu}}\right|<\omega\frac{(x-\xi)^{n-\nu}}{(n-\nu)!} \quad (\nu=1,2,\ldots,n-1). \quad . \quad . \quad (3)$$

Let us now consider an arbitrary real function $\psi(x)$, which is $n$ times differentiable in the interval $\xi \leq x \leq \xi+\varepsilon$, which satisfies the differential equation in the interval $\xi < x \leq \xi+\varepsilon$ and which satisfies the initial conditions. My object in writing this article is to prove $\psi(x)=\varphi(x)$ for $\xi \leq x \leq \xi+\varepsilon$.

If the positive number $\gamma \leq \varepsilon$ is small enough, the inequalities (3) hold in the interval $\xi < x \leq \xi+\gamma$ with $\psi(x)$ for $\varphi(x)$. Let $\delta$ be the upper bound of these numbers $\gamma$.

It is sufficient to show that $\varphi(x)=\psi(x)$ in the interval $\xi \leq x \leq \xi+\delta$. Indeed, I assert that in that case $\delta=\varepsilon$, so that $\varepsilon$ possesses the required properties. In fact, suppose $\delta<\varepsilon$. Since $\psi(x)=\varphi(x)$ in the interval $\xi \leq x \leq \xi+\delta$, it follows from considerations of continuity that (3) holds also in the vicinity of $\delta$, contrary to the choice of $\delta$. Hence $\delta=\varepsilon$.

The functions

$$(x-\xi)^{-n+\nu-1}\left(\frac{d^{\nu-1}\psi(x)}{dx^{\nu-1}}-\frac{d^{\nu-1}\varphi(x)}{dx^{\nu-1}}\right) \qquad (\nu=1,2,\ldots,n)$$

tend with $x-\xi$ to zero. Hence

$$M=\max_{\substack{1\leq\nu\leq n\\ \xi<x\leq\xi+\delta}}\frac{(n-\nu+1)!}{(x-\xi)^{n-\nu+1}}\left|\frac{d^{\nu-1}\psi(x)}{dx^{\nu-1}}-\frac{d^{\nu-1}\varphi(x)}{dx^{\nu-1}}\right|$$

is a finite number. Since $\varphi(x)$ and $\psi(x)$ satisfy the differential equation, we have in the interval $\xi < x \leq \xi+\delta$

$$\left|\frac{d^{n}\psi(x)}{dx^{n}}-\frac{d^{n}\varphi(x)}{dx^{n}}\right|$$

$$\leq\left|f\left(x,\psi,\frac{d\psi}{dx},\ldots,\frac{d^{n-1}\psi}{dx^{n-1}}\right)-f\left(x,\varphi,\frac{d\varphi}{dx},\ldots,\frac{d^{n-1}\varphi}{dx^{n-1}}\right)\right|.$$

From (1) it follows that in the interval $\xi \leq x \leq \xi+\delta$

$$\left|\frac{d^{n}\psi(x)}{dx^{n}}-\frac{d^{n}\varphi(x)}{dx^{n}}\right|\leq M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and therefore

$$\left|\frac{d^{n-1}\psi(x)}{dx^{n-1}}-\frac{d^{n-1}\varphi(x)}{dx^{n-1}}\right|\leq M(x-\xi) \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It is sufficient to show $M=0$.

Let us first consider the case in which in (5) the equality holds for every $x>\xi$ in the vicinity of $\xi$.

The left-hand side being a continuous function of $x$, we have for these numbers $x$

$$\frac{d^{n-1}\psi(x)}{dx^{n-1}}-\frac{d^{n-1}\varphi(x)}{dx^{n-1}}=M(x-\xi) \text{ or } -M(x-\xi),$$

which implies

$$\frac{d^n \psi(x)}{dx^n} - \frac{d^n \varphi(x)}{dx^n} = M \text{ or } -M \text{ for } x = \xi;$$

hence $M = 0$, since

$$\frac{d^n \varphi(x)}{dx^n} \text{ and } \frac{d^n \psi(x)}{dx^n}$$

assume at $\xi$ the same value $\eta_{n+1}$.

Now it is sufficient to show that the remaining case is excluded. Let $x$ be an arbitrary number $> \xi$ and $\leqq \xi + \delta$. In the remaining case there would exist a number $\lambda > \xi$ and $< x$ satisfying the inequality

$$\left| \frac{d^{n-1} \psi(\lambda)}{d\lambda^{n-1}} - \frac{d^{n-1} \varphi(\lambda)}{d\lambda^{n-1}} \right| < M(\lambda - \xi)$$

and from (4) it would follow that

$$\left| \frac{d^{n-1} \psi(x)}{dx^{n-1}} - \frac{d^{n-1} \varphi(x)}{dx^{n-1}} - \frac{d^{n-1} \psi(\lambda)}{d\lambda^{n-1}} + \frac{d^{n-1} \varphi(\lambda)}{d\lambda^{n-1}} \right| \leqq M(x - \lambda).$$

Hence (5) would follow with $<$ for $\leqq$ and by repeated integration we should find

$$\left| \frac{d^{\nu-1} \psi(x)}{dx^{\nu-1}} - \frac{d^{\nu-1} \varphi(x)}{dx^{\nu-1}} \right| < M \frac{(x-\xi)^{n-\nu+1}}{(n-\nu+1)!} \qquad (\nu = 1, \ldots, n; \xi < x \leqq \xi + \delta),$$

which is impossible, since according to the definition of $M$ the two sides are equal one to another for conveniently chosen $\nu$ and $x$ ($\xi < x \leqq \xi + \delta$). This establishes the theorem.

**Theorem 2.** *Consider* $n+2$ *real numbers* $\xi, \eta_1, \ldots, \eta_{n+1}$, *further a positive number* $\omega$ *and the polynomial*

$$p(x) = \sum_{\nu=0}^{n} \eta_{\nu+1} \frac{(x-\xi)^\nu}{\nu!}.$$

*Let the real function* $f(x, y_1, \ldots, y_n)$ *be defined in the* $(n+1)$ — *dimensional strip*

$$\xi < x < \xi + \omega, \ -\infty < y_1 < \infty, \ldots, -\infty < y_n < \infty.$$

*Suppose that in the region*

$$\xi < x < \xi + \omega, \qquad \left| y_\nu - \frac{d^\nu p(x)}{dx^\nu} \right| < \omega \frac{(x-\xi)^{n-\nu}}{(n-\nu)!} \qquad (\nu = 1, \ldots, n)$$

*the derivatives* $f_\nu = \frac{\partial f}{\partial y_\nu}$ *of the first order of* $f$ *exist, are continuous and satisfy the inequality*

$$\sum_{\nu=1}^{n} |f_\nu(x, y_1, \ldots, y_n)| \frac{(x-\xi)^{n-\nu+1}}{(n-\nu+1)!} \leqq 1.$$

*Then the assertion of theorem 1 is valid.*

In fact, we have

$$|f(x, Y_1, \ldots, Y_n) - f(x, y_1, \ldots, y_n)| = \left| \sum_{\nu=1}^{n} \frac{\partial f(x, \lambda_1, \ldots, \lambda_n)}{\partial \lambda_\nu} \cdot (Y_\nu - y_\nu) \right|.$$

where $(\lambda_1, \ldots, \lambda_n)$ is a point conveniently chosen between $(y_1, \ldots, y_n)$ and $(Y_1, \ldots, Y_n)$ and the found expression is

$$\leqq \left\{ \max_{1 \leqq \nu \leqq n} \frac{(n-\nu+1)!}{(x-\xi)^{n-\nu+1}} |Y_\nu - y_\nu| \right\} \sum_{\nu=1}^{n} \left| \frac{\partial f(x, \lambda_1, \ldots, \lambda_n)}{\partial \lambda_\nu} \right| \cdot \frac{(x-\xi)^{n-\nu+1}}{(n-\nu+1)!}$$

$$\leqq \max_{1 \leqq \nu \leqq n} \frac{(n-\nu+1)!}{(x-\xi)^{n-\nu+1}} \cdot |Y_\nu - y_\nu|,$$

so that theorem 2 is a corollary of the first proposition.

**Mathematics.** — *Die Kovarianten von vier Ebenen im $R_5$.* Von R. WEITZENBÖCK.

(Communicated at the meeting of January 31, 1942.)

Ich habe in einer früheren Arbeit [1]) die projektiven Invarianten von vier Ebenen im $R_5$ ermittelt; hier sollen die Kovarianten mit einer Reihe Punktkoordinaten $x$ berechnet werden. Man findet deren acht, alle vom dritten Grade in den $x$, die durch drei quadratische Syzygien verknüpft sind. Zugleich ergibt sich, dass drei Ebenen des $R_5$ keine Kovarianten mit nur einer Reihe und daher auch keine Kontravarianten mit nur einer Reihe $R_4$-Koordinaten $u'$ besitzen.

§ 1.

Wir deuten die vier Ebenen durch $a$, $\alpha$, $p$, $\pi$ oder 1, 2, 3, 4 an. Der Aufbau der Komitanten führt dann, genau so wie in obgenannter Arbeit, auf Klammerfaktoren der Typen

$$f_1 = (a^3\,\alpha^2\,p) \text{ und } f_2 = (a^3\,\alpha^2\,x).$$

Hier leitet $f_1$ zu Ketten der Gestalt

$$f_3 = (a^3\,\alpha^2\,p)\,(p^2\,\pi^3\,b)\,(b^2\,\beta^3\,q)\ldots = 12 \cap 34 \cap 12 \cap \ldots,$$

die auf Invarianten

$$A_{ik} = (i^3\,k^3) \text{ und } J_{ijkl} = (i^3\,j^2\,k)\,(jk^2\,l^3)$$

reduzierbar sind.

Bei $f_3$ setzen wir

$$P_{ik} = (i^2\,k^3\,x)\,i, \qquad (1)$$

was also neben der Reihe $x$ die sechs Reihen

$$P_{12}, P_{13}, P_{14}, P_{34}, P_{42}, P_{23}$$

gibt. Es gilt dann

$$P_{ik} = P_{ki} - \tfrac{1}{3}\,A_{ik}\,x. \qquad (2)$$

Geometrisch ist $P_{ik}$ der in der Ebene $E_i$ gelegene Punkt des $R_3$, der $x$ mit $E_k$ verbindet. Es gilt also z.B.

$$(1^3\,P_{1i}\,\xi\,\eta) = 0\ ,\ (1^3\,P_{i1}\,\xi\,\eta) = -\tfrac{1}{3}\,A_{i1}\,(1^3\,x\,\xi\,\eta)$$

und

$$(1^3\,P_{1i}\,x\,\xi) = (1^3\,P_{i1}\,x\,\xi) = 0.$$

Nimmt man in (1) statt $x$ ein $P_{rs}$ selbst, so ergeben sich die Reihen

$$(i^3\,k^2\,P_{rs})\,k$$

[1]) Proc. Kon. Akad. v. Wetensch., Amsterdam, **35**, 1026—1029 (1932).

und dies ist nur dann nicht unmittelbar zu reduzieren, wenn $i, k, r$ und $s$ alle vier Ziffern 1, 2, 3, 4 bedeuten. Man erhält so sechs weitere Reihen

$$Q_{ik} = (i^2\, k^3\, P_{rs})\, i \quad \ldots \ldots \ldots \quad (3)$$

oder ausführlicher:

$$Q_{12} = (1^2\, 2^3\, P_{34})\, 1 \;,\; Q_{13} = (1^2\, 3^3\, P_{42})\, 1 \;,\; Q_{14} = (1^2\, 4^3\, P_{23})\, 1 \;\text{ u. s. f.}$$

Es gilt analog zu (2):

$$Q_{12} = Q_{21} - \tfrac{1}{3} A_{12} \,.\, P_{34} . \quad \ldots \ldots \ldots \quad (4)$$

Verfährt man mit $Q_{ik}$ ebenso wie in (3) mit den $P_{ik}$, so ergeben sich Reihen der Gestalt

$$\left. \begin{aligned} R_{12} &= (1^2\, 2^3\, Q_{34})\, 1 = (1^2\, 2^3\, 3)\, (3^2\, 4^3\, P_{12})\, 1 = \\ &= (1^2\, 2^3\, 3)\, (3^2\, 4^3\, a)\, (a^2\, \alpha^3\, x)\, 1 = 1 - 12 \cap 34 \cap 12 - x, \end{aligned} \right\} \quad \ldots \quad (5)$$

also reduzible Ketten.

Es ist dann weiter leicht zu zeigen, dass jeder Ansatz

$$J = (a^3\, \alpha\, \xi\, \eta) \;\text{ mit }\; \xi, \eta = x, P_{ik}, Q_{rs}$$

entweder zur Einführung einer weiteren Reihe $P_{ik}$ oder zur Reduktion führt. Somit bleiben nur die Typen

$$J = (a^3\, \xi\, \eta\, \zeta) \;\text{ mit }\; \xi, \eta, \zeta = x, P_{ik}, Q_{rs} \quad \ldots \ldots \quad (6)$$

und die sechsreihigen Determinanten übrig, die man aus sechs der Reihen $x$, $P_{ik}$ und $Q_{rs}$ bilden kann.

## § 2.

Was zunächst diese sechsreihigen Determinanten betrifft, so sieht man leicht, dass sie auf die Invarianten (6) zurückführbar sind. Wir haben z.B.

$$(P_{12} \ldots) = (1 \ldots)\, (1^2\, 2^3\, x)$$

und bringt man hier alle drei Reihen 1 in den ersten Klammerfaktor, so entstehen Produkte von Invarianten des Typus (6). Analog für Determinanten, die eine Reihe $Q$ enthalten.

Wir führen dies bei der Determinante

$$D = (P_{12}\, P_{13}\, P_{14}\, P_{34}\, P_{42}\, P_{23}) = (1\, P_{13}\, P_{14}\, P_{34}\, P_{42}\, P_{23})\, (1^2\, 2^3\, x) \qquad (7)$$

näher aus und erhalten

$$D = \tfrac{1}{3} (1^3\, P_{23}\, P_{34}\, P_{42}) \,.\, (2^3\, x\, P_{13}\, P_{14}) = \tfrac{1}{3} F_1 \,.\, G_1 . \quad \ldots \quad (8)$$

Für die Invarianten (6), bei denen keine Reihe $Q_{ik}$ auftritt, führen wir die folgenden Bezeichnungen ein:

$$\left. \begin{aligned} F_1 &= (2^3\, x P_{13}\, P_{14}) & G_1 &= (1^3\, P_{23}\, P_{34}\, P_{42}) \\ F_2 &= (3^3\, x P_{24}\, P_{21}) & G_2 &= (2^3\, P_{34}\, P_{41}\, P_{13}) \\ F_3 &= (4^3\, x P_{31}\, P_{32}) & G_3 &= (3^3\, P_{41}\, P_{12}\, P_{24}) \\ F_4 &= (1^3\, x P_{42}\, P_{43}) & G_4 &= (4^3\, P_{12}\, P_{23}\, P_{31}) \end{aligned} \right\} \quad \ldots \quad (9)$$

Die Bezeichnung ist dabei so gewählt, dass $F_i$ aus $F_{i-1}$ durch zyklische Vertauschung

der Ziffern 1, 2, 3, 4 entsteht; ebenso bei $G_i$. Jede dieser Kovarianten ist vom dritten Grade in den $x_i$. So lauten z.B. $F_1$ und $G_1$ ausgeschrieben

$$F_1 = (2^3\,xab)\,(a^2\,p^3\,x)\,(b^2\,\pi^3\,x)$$
$$G_1 = (1^3\,234)\,(2^2\,p^3\,x)\,(3^2\,\pi^4\,x)\,(4^2\,\alpha^3\,x),$$

woraus die geometrische Bedeutung von $F_1 = 0$ und $G_1 = 0$ leicht abzulesen ist.
Man zeigt weiter leicht, dass alle weiteren Komitanten (6), wie z.B. $(a^3\,xPQ)$ oder $(a^3\,QQQ)$ u.s.w. auf die Kovarianten (9) reduzierbar sind. Da ferner

$$F_1 = (2^3\,xP_{13}\,P_{14}) = -(3^3\,xP_{12}\,P_{14}) = -(2^3\,xP_{14}\,P_{13}), \quad . \quad (10)$$

also in den Indizes 2, 3, 4 alternierend ist, folgt, dass jede Kovariante auf die acht $F_i$ und $G_i$ von (9) reduzierbar wird.

## § 3.

Dass von den acht Kovarianten (9) keine ganz und rational durch die übrigen ausdrückbar ist, folgt aus den Graden dieser Kovarianten in den Koordinaten der vier gegebenen Ebenen. Dagegen sind diese Kovarianten (9) durch quadratische Syzygien verknüpft, die man am einfachsten wie folgt erhält.

Wir gehen von (7) aus und schreiben

$$D = (P_{42}\,P_{12}\,P_{13}\,P_{14}\,P_{34}\,P_{23}) = (\pi P_{12}\,P_{13}\,P_{14}\,P_{34}\,P_{23})\,(\pi^2\,\alpha^3\,x).$$

Hier bringen wir alle drei Reihen $\pi$ in den ersten Klammerfaktor, wodurch sich nach einiger Rechnung

$$D = \tfrac{1}{3}\,F_4\,G_4 + \tfrac{1}{9}\,F_2\,(A_{34}\,F_1 + A_{14}\,F_3 + A_{31}\,F_4) \quad . \quad . \quad . \quad (11)$$

ergibt. Gleichsetzung mit (8) und nachherige zyklische Vertauschung liefert die gesuchten Syzygien:

$$\left.\begin{aligned}
S_1 &= F_1\,G_1 - F_4\,G_4 - \tfrac{1}{3}\,(A_{34}\,F_1\,F_2 + A_{14}\,F_2\,F_3 + A_{31}\,F_2\,F_4) = 0\\
S_2 &= F_2\,G_2 - F_1\,G_1 - \tfrac{1}{3}\,(A_{41}\,F_2\,F_3 + A_{21}\,F_3\,F_4 + A_{42}\,F_3\,F_1) = 0\\
S_3 &= F_3\,G_3 - F_2\,G_2 - \tfrac{1}{3}\,(A_{12}\,F_3\,F_4 + A_{32}\,F_4\,F_1 + A_{13}\,F_4\,F_2) = 0\\
S_4 &= F_4\,G_4 - F_3\,G_3 - \tfrac{1}{3}\,(A_{23}\,F_4\,F_1 + A_{43}\,F_1\,F_2 + A_{24}\,F_1\,F_3) = 0
\end{aligned}\right\} . \quad (12)$$

Hier ist $S_1 + S_2 + S_3 + S_4$ identisch in $F_i$ und $G_i$ Null, d.h. es sind in (12) höchstens nur drei unabhängige Gleichungen vorhanden.

Die Frage nach den Kontravarianten mit einer Reihe $R_4$-Koordinaten $u'$ ist hiermit ebenfalls erledigt. Man erhält dual zu (12) wieder acht Komitanten dritten Grades, z.B.

$$F_1' = (2'^3\,u'\,P_{13}'\,P_{14}') \text{ mit } P_{13}' = (1'^2\,3'^3\,u')\,1' \text{ und } P_{14}' = (1'^2\,4'^3\,u')\,1' \text{ u. s. f.}$$

Für die Invarianten gilt:

$$A_{12}' = (1'^3\,2'^3) = A_{12} = (1^3\,2^3)$$
$$J_{1234}' = J_{1234} = (1^3\,2^2\,3)\,(23^2\,4^3) = (1'^3\,2'^2\,3')\,(2'\,3'^2\,4'^3).$$

**Botany.** — *Contact prints of wood.* By L. G. M. BAAS BECKING and JOHA. WALENKAMP.
(From the Botanical Institute, Government University, Leiden.)

(Communicated at the meeting of January 31, 1942.)

In the excavations performed under the direction of Prof. Dr. A. E. VAN GIFFEN several specimens of wood, in various stages of preservation, have been brought to light. In order to obtain a permanent record of the characteristics of those samples, the handling of which is often cumbersome, a method was developed which enabled us to obtain many prints from a single sample of wood. The method described in this note has been applied chiefly to cross-sections. Although tangential and radial sections yielded promising results, the preparation of contact prints on these planes has not yet been fully worked out.

1. Oak wood (Quercus sessiliflora Sm.).

Cross sections of logs were made by means of a saw and those sections were carefully planed. From such a freshy planed section contact-prints cannot be obtained. After 3—4 days drying sufficient relief appeared. Observing the section with a hand lens at a glancing angle showed the vessels protruding from the rest of the tissue. This differentiation of high and low relief is brought about by the differences in the directions of maximal swelling, which is perpendicular to the direction of the micellae. Observation of thin radial sections under polarized light corraborated this supposition.

Shrinkage of wood-parenchyma and medullary-ray cells rather than changes in the longitudinal dimension of the vessels, therefore, caused the surface-differentiation necessary to prepare contact-prints.

Plaster casts of the above-mentioned surface showed beautiful details, the individual vessels (appearing as pits) being clearly visible by means of a hand lens.

If, on the contrary, dry oak wood is cross-sawed and planed and afterwards soaked in boiling water for 15—20 minutes, a differentiated surface appears which is, in a great many respects, a counter-mould of the first-mentioned cross-section. The vessels appear either flush with the surface, but mostly sunken. Medullary rays and parenchyma appear in high relief. A plaster cast of such a surface shows the vessels as rings, while much detail may be oberved of both ray- and other parenchymatous tissue.

Prints were made of the above preparations by either inking the surface, covering it with a suitable paper and hammering the top-side of the paper with a felt-covered hammer, or the ink was applied on the top-side of the paper, and "hammered-through".

Inking of the wood-surface yielded the best results. We used the following inks and stains; printers ink, mimeograph ink, copy-ink, Indian ink, Azure-blue, Fuchsin and Nigrosin-black. The Nigrosin-black proved to be the most suitable. The ink has to be almost dry before a print can be made. The paper which proved to be most suitable was a very thin type writer copy-sheet. Cellophane, however, yielded the most superior results, if its surface is free from grease.

Collodion films prepared on top of the inked surface gave trouble because of the many air bubbles escaping from the lumina of the vessels and so rupturing the film.

Figure 1 shows a paper-nigrosin print of a cross section of oak wood, obtained from the foundation of a Roman Castellum, excavated near Valkenburg S. Holland and marked *520.* This wood was sawed when wet and allowed to dry afterwards.

Photo 1 shows a part of this print, enlarged 15 times. Much anatomical detail is visible.

Photo 2 is made from a cellophane print, enlarged 15 times. The wood (recent oak) was sawed when dry and soaked in boiling water afterwards. Much more detail is visible in this print, which is, in many respects, a countermould of Photo 1.

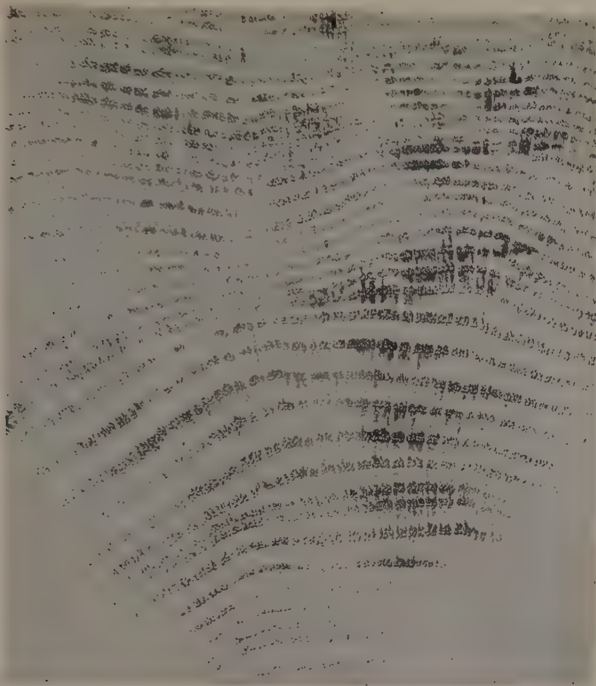

Fig. 1.

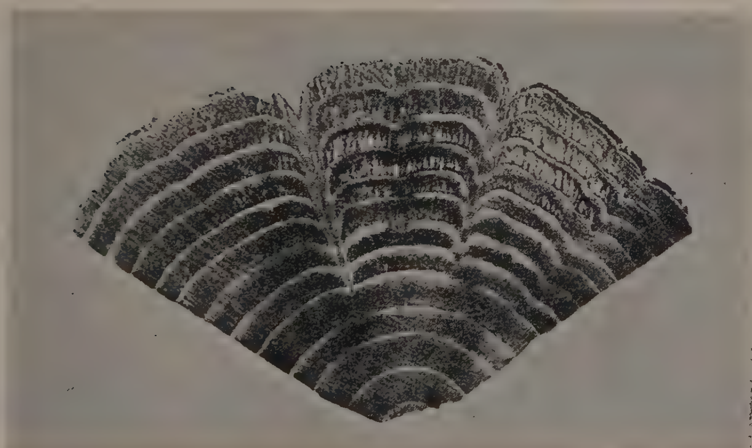

Fig. 2.

Photo 1.

Photo 2.

2. *Other, so called "ring-porous" wood* yields suitable prints.

Both elm and ash, recent and excavated, were used.

3. *Coniferous wood.*

One might expect scanty detail in contact prints, due to the monotonous anatomical structure. To our surprise, however, good prints could be obtained. Figure 2 shows a nigrosin-paper print of Pinus radiata Don., the Monterey-pine. The spring-wood takes the ink, the closer-built and more resinous autumn-wood fails to take the stain (Figure 2). With several other conifers, e.g. CEDRUS, ABIES and TAXUS, similar results were obtained. In some cases an effective counterstain could be found in Scarlet red, which stained the autumn wood. Two-colour prints were made in this way.

Further trials are needed to perfect the method outlined in this note. It seems possible, in collaboration with a wood-technologist, to elaborate a field method, by which a complete record could be obtained from the stems of a felled parcel of forest, the records of which may yield valuable information both to the ecologist and to the forester.

Medicine. — *On hypoproteinemia.* By E. GORTER.

(Communicated at the meeting of December 27, 1941.)

My interest in the problem of hypoproteinemia was roused by a new method of investigation, in which use is made of the property which many substances have, such as fats, lipoids, amines, alcohols, but also proteins, of easily spreading in a monomolecular layer of constant thickness and dimensions.

When we are in possession of an apparatus to determine the area at varying pressure, it is easy to determine the area occupied by a certain small quantity of protein. In this way it can also be easily determined what the area is of a certain quantity of a serum spread in a monomolecular layer.

When the circumstances are well chosen, and the maximal spreading is measured, which is obtained by placing 0.1 n HCl in the tray, then easily reproducible results are obtained. In order to reduce the square metres found to milligrams the method should be tested by examining how much area is occupied by one milligram in a monomolecular layer. We have lately done this again together with Ir. P. C. BLOKKER [1]). It was then seen, that

1 mg serum-albumin spreads $1.04 \times 10^4$ cm$^2$.

1 mg serum-globulin spreads $0.93 \times 10^4$ cm$^2$.

By globulin we have always meant the protein, which is precipitated when a serum is half saturated with ammonium sulphate.

With the spreading method it is not possible to determine fibrinogen which occurs in plasm but not in serum, because it does not spread unless a small quantity of trypsin is added.

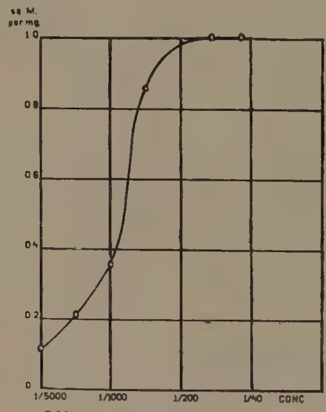


Fig. 1.

It can now be easily understood how serum proteins are determined. Blood is taken from the patient under certain precautions to prevent contamination with spreading substances. The syringe is boiled in water free from fat, the skin is cleaned with alcohol and ether, the glassware is made free from fat with bichromate and sulphuric acid. The blood is coagulated and after some time the serum is collected by centrifuging.

0.1 cm$^3$ of this serum diluted 10 times is measured and this dilution serves to determine all proteins (albumin as well as globulin). In another tube 0.1 cm$^3$ is pipetted and then 0.1 cm$^3$ of a saturated watery solution of ammonium sulphate is added. There is a precipitation of the globulins. This precipitate is washed with a half-saturated solution

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[1]) E. GORTER and P. C. BLOKKER, Proc. Ned. Akad. v. Wetensch., Amsterdam, **45**, 151 (1942).

of ammonium sulphate, and finally dissolved in 1 cm³ water. This solution is used to determine the percentage of the globulins. For the determination of the protein in these solutions the tray of the apparatus of LANGMUIR is filled with 1 n HCl. After cleaning 5 mm³ of the protein solution is blown out at the watersurface from a capillary pipette. The protein then spreads in a layer of ca. 10 Å thickness over part of the area, it is spread maximally. This area may be measured and determined while placing it under increasing pressure. Thus we obtain a pressure area curve, from which by extrapolation to pressure zero we determine the area under zero pressure. From the figures found we calculate the area of 1 cm³ of the protein solution. It is only necessary to divide by the factor which indicates how many square metres is occupied by 1 mg of the protein examined. The albumin is found by deducting the area of the globulins from that of the total proteins.

The problem of hypoproteinemia can be studied by this or by a different method. The spreading method has the advantage that it gives exact results, even when the quantity of blood is very small. It stands to reason that especially the children's specialist and the patient appreciate this advantage.

*Clinic of hypoproteinemia.*

Now I must at once place you in medias res and will do this by reading to you a number of case histories of patients in whom we found hypoproteinemia. This enables me to illustrate why as medical men we are interested in this problem and by pointing out to you the great differences between the diseases characterised by a loss of blood proteins. I can make it clear that the examination must extend to the entire patient. For the sake of clarity I have collected some figures concerning the patients in tables, but what is communicated about etiology, course and details of the diseases is not made superfluous.

The first patient has the picture of hypoproteinemia in its simplest form. It is a boy of 2½ years, his complaints are a puffy face, swollen arms, legs and abdomen. He has gained much in weight. He urinates less than usual, smaller quantities and less often. He is listless and feels ill, possibly he is feverish. His appetite is poor. He has vomited much and complains of thirst.

These symptoms have existed only 5 days. He has not been ill before. He has only coughed since the beginning of February, following a bad cold. The cough resembled whooping cough a little.

The family is healthy, he is the second of three children. The diet has been a little one sided. It consisted principally in bread, rusks and much vegetables, with little milk, little meat, very little fruit and not always potatoes.

During his first year he was given cod liver oil. Although it is not possible to make quite sure, the food seems especially to have been poor in proteins, and vitamins A and B have probably been scanty.

Examination shows a puffy face, praetibial edema, a strongly inflated abdomen, so considerable edema. This will soon disappear on a saltless diet, while in 12 days the bodyweight decreases from 13.8 kg to 11.04 kg.

On examination of the heart there is not a single symptom of insufficiency of the heart muscle. The liver is not enlarged, there is no abnormality in the urine, there is no protein, the sediment does not contain cells or crystals.

The WASSERMANN reaction is negative.

Protein in Blood.

| Data | Alb. | Glob. | Fibrin. | Total | Ureum | Cholesterol |
|---|---|---|---|---|---|---|
| 25/4 | 3.0 % | 1.1 % | 0.37 % | 4.5 % | 260 mg/l | 172 mg % |
| 3/5 | 3.5 % | 1.2 % | — | 5 % | | |
| 7/5 | 4.0 % | 1.4 % | — | 5.5 % | | |

Haem. 13.3 gr %; erythr. 4.700.000.

We have considered this case as hypoproteinemia with the usual consequences, caused by insufficient proteinfeeding. In young babies this hypoproteinemia is frequently found with insufficient feeding.

The protein percentage is much too low: 4.5 % (instead of at least 6 %), but it increases rapidly to 5.5 %. The decrease of the globulin is greater than of the albumin. The hypoproteinemia is not combined with high figures for cholesterol and lipoids. The kidneys function normally: the ureum of the blood is low.

I will now tell you about a case in which the decrease of the proteins was caused by considerable albuminuria.

The girl, 3½ years of age, became ill in the early part of May with fever. It had also been noticed that her face was swollen and that the urine was brown. Her legs had also become swollen. For some months she has been in a local hospital, but had improved little. She had never been ill before, and the family is also healthy. Her diet cannot be considered as deficient. As cause of the symptoms we find a very great quantity of protein in the urine.

Protein in Blood.

| Data | Alb. | Glob. | Total | Ureum | Rest N | Cholesterol | Lipoids |
|---|---|---|---|---|---|---|---|
| 3 Dec. | 2.9 | 1.7 | 4.6 % | 180 mg % | | 421 mg % | 1.76 g % |
| 9 Dec. | 2.7 | 1.6 | 4.3 % | | | | |

| Data | K | Na | Chloride | Ca | P |
|---|---|---|---|---|---|
| 11 Dec. | 15.2 | 320.6 | 421 | 7.8 | 4.5 mg % |

You recognize some of the phenomena found in the first patient, and seen almost regularly in hypoproteinemia.

In these cases of nephrosis — a degenerative kidney disease — the figures for the total protein have much decreased. Mostly it is a decrease of albumin. Moreover we see a strong increase of the cholesterol and the lipoids of the blood as additional symptoms.

I now arrive at a third group of diseases in which hypoproteinemia is found with its consequences, but the cause of which is not known.

Because of this the picture has been called essential hypoproteinemia by COPE and GOADBY [1]).

The first case they call by this name concerns a man of 20, with edema, without albuminuria, with a urea-clearance of 70 %.

Protein in blood: albumin 3.13, globulin 1.61. Total 4.6 %.

Protein in Blood.

| Diet | 1934 | Alb. | Glob. | Fibrin. | Tot. | Rest. N | Ureum | Ca | P |
|---|---|---|---|---|---|---|---|---|---|
| | —/5 | 3.13% | 1.51% | — % | 4.6 % | — | | | |
| Low protein | 8/6 | 2.90% | 1.69% | 0.45% | 4.58% | 23 | 250 mg/l | 10.1 | 4.1 |
| High protein | 18/6 | 2.67% | 1.89% | 0.80% | 4.56% | 40 | 560 | 8.8 | 5.0 |
| Liver | 24/6 | 2.76% | 1.51% | 0.66% | 4.26% | 44 | | 8.9 | 3 5 |
| Normal | 4/8 | 1.94% | 1.77% | — % | 3.70% | 37 | | | |
| Normal | 17/9 | 2.50% | 1.64% | — % | 4.10% | | | | |

Our patient whose case we have called by the same name is a girl of 10.

This girl has frequently a swollen face, legs and abdomen. When she rests and gets

1) LANCET, 1935, May 4.

little salt she urinates much and the edema disappears. It returns when she exerts herself. Otherwise the child does not feel ill and there is no other symptom of disturbed heart or kidney functions. She has never before had such disturbances. Of infantile diseases she has only had measles, whooping cough and chickenpox, otherwise she has not been ill either.

This returning edema has been observed in the child from the age of 3 years onwards. No such cases of edema occur in the rest of the family.

On examination we find considerable edema of the entire body. She has a puffy face, praetibial edema, ascites. There is also marked swelling of the coniunctiva bulbi.

To our surprise there is no indication of disturbed kidney function. The urine never contains protein, there is no sediment. We looked in vain for any symptom of insufficientia cordis. The bloodpressure is 105/85. The ureum and the rest N of the blood is low. The urea clearance is 63 %. The quantity of cholesterol is low, that of the lipoids is normal. The inorganic composition of the blood is also normal.

Protein in Blood

| Per. | Date | Alb. | Glob. | Fibr. | Tot. | Ureum | Rest N | Cholesterol | Lipoids |
|---|---|---|---|---|---|---|---|---|---|
| I | 30/9 | 3.9 | 1.3 | 0.38 | 5.2% | | | | |
| | 1/10 | — | — | — | — | 390 mg L. 360 | 36.7 mg% | 90 mg % | 593 mg % |
| II | 6/10 | 3.5 | 1.2 | — | 4.7% | | | | |
| VI | 7/11 | 3.8 | 1.2 | — | 5.0% | | | | |
| VII | 14/11*) | 3.7 | 1.0 | — | 4.7% | | | | |
| VII | 17/11 | 3.6 | 1.2 | — | 4.8% | | | | |
| VII | 18/11*) | 3.3 | 1.1 | — | 4.4% | | | | |
| | 4/12 | 4.0 | 1.1 | — | 5.1% | 320 mg L. | 31 mg % | 146 mg % | 562 mg % |
| VIII | 18/12 | 3.6 | 1.2 | — | 4.8% | | | | |

| Per | Date | K | Na | Chlorides | Date | Ca | P |
|---|---|---|---|---|---|---|---|
| I | 1/10 | 17.7 | 368 mg % | 398 mg % 427 mg % | 29/9 | 8 mg % | 3.1 mg % |
| | 11/12 | 15.8 | 316 mg % | 424 mg % | | | |

*) Injection lyoph. serum 4 and 6 cc: Hem. 14.5 gr %; Erythroc. 4.370.000.

The only abnormality is a decrease of the protein percentage of the blood. There is no anemia: hemoglobin 14.5 g %. Erythrocytes 4.370.000.

Course. During her stay in the clinic we can confirm the observation of the parents that under the influence of rest and a diet with little salt the edema disappears rapidly, but that it rapidly returns when salt is given. We have also ascertained, that a diet rich in protein does not improve the illnes.

*Epricrisis.*

As noteworthy features of this case of hypoproteinemia we would mention the following facts: absence of any symptom of a disease of heart or kidneys, absence of the increase of cholesterol and lipoids in the blood serum and of the increase of ureum and rest nitrogen, and especially the pertinacity with which the low percentage of proteins in the blood is maintained in spite of the high protein percentage of the diet and the injection of lyophile serum in the blood. Moreover it is striking that the decrease of the protein is especially a decrease of globulin, to a less extent of albumin, so that the proportion albumin : globulin is usually 3 : 1. This proportion also changes very little. The fibrinogen percentage is normal.

Recapitulating the course of the illness: the edema has existed from the age of three years and is unchanged, the girl is now 10 years old. There has been no previous illness

| | Diet | | |
|---|---|---|---|
| | of Periods II [1]) — V | of Periods VI — VII — VIII | |
| Eggs | (1) | (1) | |
| Sugar | $37^1/_2$ gr | $37^1/_2$ gr | |
| Butter | 24 gr | 24 gr | |
| Bread | 200 gr | 240 gr | |
| Rice | 45 gr | 45 gr | |
| Gravy | 18 gr | 18 gr | |
| Meat | $37^1/_2$ gr | 75 gr | |
| Vegetables | — | — | |
| Potatoes | 200 (250) [1]) gr | 250 gr | |
| Fruit | 50 gr | 100 gr | |
| Red currant juice | 50 cc | 50 cc | |
| Gelatin | — | 5 gr | |
| Liver | — | 100 gr | |
| Curds [2]) | — | 150 gr | 300 gr |
| Serum of oxen [3]) | — | — | 60 cc |

and intercurrent diseases such as measles, whooping cough and chickenpox have left no traces.

From all this we conclude that there is essential hypoproteinemia. The disease does not occur in the family, both father and mother have a normal protein percentage.

From this and many other clinical observations we can at least deduce, what will be the consequences of hypoproteinemia.

*The consequences of hypoproteinemia.*

Of one symptom the edema, it is certain that it is caused by hypoproteinemia.

It appears that the chance of it is the greater as the serum albumin is lowered. All the other consequences of hypoproteinemia, except increased susceptibility for certain infections, are doubtful. The influence of protein in the bloodplasm and the great influence of serum albumin in the production of edema are accounted for by the osmotic pressure, depending on the proteins, being lowered. Normally there is a certain proportion between the hydrostatic pressure in the small vessels and the osmotic pressure of the plasm. Owing to this some liquid is pressed out in the arterial part of the capillary vessels, as here the bloodpressure is higher than the osmotic pressure of the proteins, whereas in the venous part of the capillaries the liquid enters the vessels, because here the blood pressure has decreased below the osmotic pressure of the proteins. In humans or in animals with hypoproteinemia this osmotic pressure is much too low, so that difficulties arise for the transition of liquid from the tissues to the blood vessels.

It is clear that the condition worsens when the liquid in the tissues also contains protein. Generally this is not the case. The difference between the osmotic pressure caused by the protein in- and outside the blood current is decisive for the liquid resorption in the venous capillaries.

That albumin has the greatest influence is because it has the smallest molecular weight.

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[1]) Not in Period II.

[2]) Contains 250 mg % NaCl.

[3]) Contains 585 mg % NaCl and 6.2 % protein, 4.2 % alb. and 2.0 % glob.

*Causes of hypoproteinemia.*

While from the case histories communicated some circumstances may be deduced which cause the loss of proteins in the blood, we can mention the following facts from experimental physiology which shed more light on this matter.

Purely theoretically our knowledge of this problem is insufficient. For we do not know with certainty where the plasm proteins are formed, although there are many arguments in favour of the liver being the principal organ for the formation. This is certain in the case of fibrinogen, the protein we shall not discuss, it is less sure for serum albumin and not very probable even for serum globulin.

The best investigations were made with drogs. They can be given hypoproteinemia by daily tapping their blood (e.g. ¼) and reinjecting the erythrocytes in salt solutions. The quantity tapped daily is regulated so that the animal constantly retains 4 gr protein per 100 cc blood plasma. It is now seen that such an animal rapidly recovers its proteins even when it gets no food. Moreover the quantity of plasm has to be tapped to keep the protein percentage low always becomes smaller until it becomes constant, see fig. *a* in J. C. MADDEN and G. H. WHIPPLE: Physiological review, **20**, 207 (1940).

This is an indication that there is a protein depot on which may be drawn. But such a dog may also be examined as to the effect of various proteins of the diet on the protein percentage. It is seen that serum protein which is given to drink is most effective, and that there are even proteins, such as zeine which cannot help to form proteins of the plasm. Perhaps this is owing to the lack of some amino acids in these proteins. An indication of this would be that some proteins inactive in these experiments, improve noticeably by the addition of one of more amino acids.

The protein percentage of a plasm can also be increased by injecting certain bacteria (pneumococci). Here it has been possible to determine quantitatively that only protein is formed which may be absorbed by the pneumococci. Soon a maximum is reached that remains constant. This increase concerns the globulins.

Can these data help us in diagnosing our patients and especially in our attempt to improve their condition?

*Treatment of hypoproteinemia.*

It is clear how a patient with starvation edema is to get rid of his hypoproteinemia. It stands to reason that this protein rapidly increases by a diet rich in proteins (3 g per kg).

It is not possible to improve the hypoproteinemia of patients in a simple way with nephrosis. The loss of protein with the urine cannot be remedied.

But for essential hypoproteinemia the improvement is more difficult still, if not impossible. We have seen the slight influence of a diet rich in proteins. Perhaps the depot has been drawn upon in this patient too and this will first be replenished. We also see that serum protein does not act much better than the other proteins. Possibly we must continue this much longer, although it will be difficult for the patient to take the liquid any longer in the great quantities required. Yet recovery may perhaps be expected. For plasm, even after concentration to ¼ volume, may be injected intravenously. This plasm must be from humans and must be dried under high vacuum at a temperature of minus 180° C. The white powder which is then obtained dissolves in a small volume of water and this concentrated serum called lyophile serum, can very well be given. Our patient too stood the injections very well. But she was given too little to expect any result. The protein has little changed in this process. Even the complement, a very labile substance, which can be kept unchanged for a week at most by preserving it below 0° can then be kept for a year. Serum dried in this way dissolves very easily, in contradistinction with serum dried in the ordinary way.

We found that when dissolved again it still spreads in exactly the same way as before the process, whereas denatured protein does not spread and will spread again only after

the addition of a small quantity of trypsin, just as the fibrinogen discussed in the introduction.

*Summary.*

Speaker gives a description of the method of determining proteins in the bloodserum by spreading in a molecular layer. He mentions the results of a series of determinations in which, together with Ir. P. C. BLOKKER he has used the spreading method and that of KJELDAHL. After this the $m^2$ found can be reduced to mg.

A number of patients have been examined by this method. As an example of various diseases in which too low a protein percentage of the blood, hypoproteinemia, was found, the author describes a case of *starvation edema*, a case of *nephrosis* and a case of essential *hypoproteinemia*. The results are given of determinations of protein, lipoid, cholesterol and inorganic substances of the bloodserum.

A summary is then given of the *consequences* of hypoproteinemia, based on clinical experience.

When the literature about *animal experiments* is consulted we find: that the *diet*, especially the sort of proteins has an evident effect on hypoproteinemia, which is the consequence of loss of plasm in dogs.

Especially serum protein, given per os, causes the blood protein percentage to increase. It also appears that any animal forms a protein depot in the tissues from which — even when it does not get food — the protein of the plasm is rapidly replaced. The blood protein does increase, but now it is the globulin that increases at different immunizations.

The treatment of hypoproteinemia begins with *restriction of the salt percentage* in the diet. When salt is given the edema reappears. A *diet* is chosen with a high protein percentage and especially serum proteins are given. Intravenously great quantities of concentrated *lyophile* serum are injected.

**Medicine.** — *Determination of serum albumin and globulin by means of spreading.* By E. GORTER and P. C. BLOKKER.

(Communicated at the meeting of December 27, 1941.)

In the laboratory of the children's hospital of the "Academisch Ziekenhuis" at Leiden, albumin and globulin have of late years been determined almost exclusively by means of spreading, as with this method there is the great advantage that the determination can be done with very little serum.

In a manner previously communicated [1]) the number of $m^2$ is determined that 1 cc of the protein solution occupies under certain circumstances on 0.1 n HCl and this figure is then reduced to the weight percentage of protein by dividing it by the so-called spreading factor, i.e. the number of $m^2$ that 1 mg protein occupies under those circumstances. 0.90 is used as spreading factor of albumin as well as of globulin. It is a well known fact that the spreading factor of nearly all proteins on 0.1 n HCl is approximately of this magnitude, e.g. 0.90 $m^2$/mg for casein, 1.00 $m^2$ for ovalbumin, 1.13 $m^2$ for haemoglobin, ca 0.90 $m^2$ for globulin and euglobulin and ca 1.04 $m^2$ for pseudoglobulin, see a.o. [2]).

In the first publication [1]) about the determination of serum globulin and albumin by means of spreading we found a spreading factor of 0.90—0.95 $m^2$/mg for albumin, but for globulin the factor was only 0.60—0.62 $m^2$/mg. After that it seemed worth while again to test the magnitude of these factors very carefully. Therefore we compared the magnitude of the spreading with the quantity of protein calculated from nitrogen determinations by the KJELDAHL method. The nitrogen determinations were made according to the micro-method described in detail by ABDERHALDEN–FODOR [3]) in which air, free from ammonia, is sucked through the solution containing the destroyed substance, and then through 0.01 n HCl.

It was found however that, contradictory to ABDERHALDEN–FODOR's instructions, it was necessary to boil the solution gently. In this way the total nitrogen percentage and the non protein nitrogen percentage of the sera examined were determined. As the separation of albumin and globulin in the spreading method was always made with ammonium sulphate and as this salt has many advantages over other salts sometimes used for this purpose, the nitrogen determinations of the globulins were also made with the globulins obtained by this method of separation. It was therefore necessary completely to remove the ammonium sulphate before the destruction. This was done by the method of CULLEN and VAN SLIJKE [4]), in which the solution is boiled with MgO and 50 % alcohol until all ammonia has disappeared. In order to be able to use as little MgO as possible, the quantity of ammonium sulphate present in the globulin obtained by the separation was determined in some cases and in further experiments more than double the amount of MgO corresponding to this quantity of ammonium sulphate was taken. Control experiments with ovalbumin solutions free from ammonium sulphate proved that the addition of ammonium sulphate had no influence on the ovalbumin nitrogen percentage obtained by the method described.

From the nitrogen percentage of the total protein and of the globulin fraction the total protein resp. globulin percentage was calculated by multiplication by 6.30. The albumin

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1) E. GORTER and F. GRENDEL, Biochem. Z., **201**, 391 (1928).

2) C. HOOFT, J. de Physiol., **36**, 652 (1938).

3) E. ABDERHALDEN and A. FODOR, Z. physiol. Chem., **98**, 190 (1917).

4) G. E. CULLEN and D. D. VAN SLIJKE, J. biol. Chem. **41**, 587 (1920).

| Human serum No. | Diagnosis | Particulars | KJELDAHL Nitrogen in gr. % | | | | Protein (6.30 × N) in gr. % | | | Spreading in $m^2/cm^3$ | | | Spreading factor in $m^2/mg$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | Total | Non protein | Tot. protein | Glo-bulin | Total | Glo-bulin | Albu-min (as diff.) | Total protein | Glo-bulin | Albu-min (as diff.) | Total | Glo-bulin | Albu-min |
| 1 | Normal | | 1.222 | 0.038 | 1.184 | 0.353 | 7.46 | 2.22 | 5.24 | 74.2 | 20.3 | 53.9 | 0.99 | 0.91 | 1.03 |
| 2 | „ | | 1.34 | 0.034 | 1.31 | 0.405 | 8.2 | 2.55 | 5.6 | 80.9 | 23.1 | 57.8 | 0.99 | 0.91 | 1.03 |
| 3 | „ | 0.94 gr. % lipoid in serum | 1.34 | 0.037 | 1.31 | 0.344 | 8.2 | 2.17 | 6.0 | 81.9 | 20.9 | 61.0 | 1.00 | 0.96 | 1.02 |
| 4 | „ | | 1.35 | 0.030 | 1.32 | 0.325 | 8.3 | 2.05 | 6.2 | 80.7 | 18.5 | 62.2 | 0.97 | 0.90 | 1.00 |
| 5 | „ | | 1.20 | 0.033 | 1.17 | 0.313 | 7.4 | 1.97 | 5.4 | 73.1 | 18.5 | 54.6 | 0.99 | 0.94 | 1.01 |
| 6 | „ | rather much lipoid in serum | 1.251 | 0.026 | 1.225 | 0.359 | 7.72 | 2.26 | 5.46 | 76.2 | 20.6 | 55.6 | 0.99 | 0.91 | 1.02 |
| 7 | leucemia ? | 8.8 mgr. % bilirubin in serum | 0.918 | 0.085 | 0.833 | 0.369 | 5.25 | 2.32 | 2.93 | 53.8 | 24.0 | 29.8 | 1.02 | 1.03 | 1.02 |
| 8 | chronic rheumatisme | sedimentation ca. 90 mm/cc | 0.99 | 0.026 | 0.96 | 0.48 | 6.1 | 3.0 | 3.1 | 61.8 | 27.4 | 34.4 | 1.01 | 0.91 | 1.11 |
| 9 | tuberculosis peritonei | | 1.284 | 0.028 | 1.256 | 0.592 | 7.91 | 3.73 | 4.18 | 74.8 | 32.7 | 42.1 | 0.95 | 0.88 | 1.01 |
| 10 | celiac disease | 0.83 gr. % lipoid in serum | 1.186 | 0.032 | 1.154 | — | 7.27 | — | — | 73.9 | — | — | 1.02 | — | — |
| 11 | acute enteritis | | 0.893* | 0.024 | 0.869 | 0.161 | 5.47 | 1.01 | 4.46 | 57.4 | 10.0 | 47.4 | 1.05 | 0.99 | 1.06 |
| 12 | tuberculosis of lung | | 0.909 | 0.032 | 0.877 | 0.414 | 5.52 | 2.61 | 2.91 | 56.8 | 24.7 | 32.1 | 1.03 | 0.95 | 1.10 |
| 13 | normal | | 1.22 | 0.027 | 1.19 | 0.32 | 7.5 | 2.0 | 5.5 | 77.0 | 18.8 | 58.2 | 1.03 | 0.94 | 1.06 |
| 14 | lipodystrophia | cholesterol in serum was normal | 1.067 | 0.028 | 1.039 | 0.275 | 6.54 | 1.73 | 4.81 | 67.2 | 15.9 | 51.3 | 1.03 | 0.92 | 1.07 |
| 15 | nephritis endocarditis | | 1.26 | 0.166 | 1.09 | 0.47 | 6.9 | 3.0 | 3.9 | 69.4 | 28.0 | 41.4 | 1.01 | 0.93 | 1.06 |
| | | | | | | | | | | | | | 1.01 | 0.93 | 1.04 |

*) According to TER MEULEN HESLINGA (Hydrating) 0.905 gr. %.

percentage was taken in the nitrogen determination as well as in the spreading as the difference between the total protein and the globulin percentage. On spreading we took no account of the lipoids in the serum, as firstly their influence can only be slight (see 1) and secondly as it is doubtful whether the spreading of a protein and that of a lipoid are additive.

The details of the method described above are as follows:

1. *Total protein.*

1 $cm^3$ serum was diluted with 1 % natrium chloride solution to 10 $cm^3$. The spreading of this was determined on 0.1 n HCl with the aid of a 5 $mm^3$ pipette. The nitrogen percentage was determined by destroying 2 $cm^3$ of the diluted serum with 1 $cm^3$ concentrated sulphuric acid (in triplicate). After the mass was dark brown 0.5 $cm^3$ 30 % peroxyde of hydrogen was added to the hot liquid, after which it was heated. This was repeated twice. After that the ammonia was made free in the apparatus of ABDERHALDEN—FODOR, and determined.

2. *Globulin.*

The separation of globulin and albumin was made by adding 1 $cm^3$ saturated ammonium sulphate to 1 $cm^3$ serum, removing the precipitated globulin by centrifuge (10 min. with ca. 4000 revolutions per min.), washing out with semi-saturated ammonium sulphate and dissolving in 1 $cm^3$ 1 % natrium chloride. This process was repeated twice. Finally the globulin solution was diluted to 10 $cm^3$ with 1% natrium chloride. The separation was made in duplicate. The spreading of the solutions obtained was determined on 0.1 n HCl with the aid of a 5 $mm^3$ pipette. The nitrogen percentage was determined by boiling twice 4 $cm^3$ of each solution with 20 $cm^3$ 50 % alcohol and 50 mg. magnesium oxide until with litmuspaper no ammonia could be detected and then destroying and determining the ammonia formed as with the total protein.

3. *Non protein nitrogen.*

1 $cm^3$ serum was shaken with 8 $cm^3$ 10 % trichloracetic acid. After 10 min. it was filtrated and the nitrogen percentage of twice 3 $cm^3$ filtrate was determined in the same way as with the total protein.

The table shows the figures obtained. In one case (total nitrogen of serum no. 11) the nitrogen percentage, besides by KJELDAHL's method, was also determined by hydrating with hydrogen and a mixture of asbestos and finely divided nickle according to TER MEULEN—HESLINGA. The figures obtained with these methods which differ fundamentally are well in accordance with each other. The total protein percentage was also determined by the rather rough method of KAGAN [5]) in which the falling time is determined of a drop of serum in a mixture of mineral oil and methyl salicylate, and from the specific gravity found the protein percentage was calculated. The protein percentages obtained by this more or less empirical method were on the average 0.5 % lower than those obtained by the KJELDAHL method.

From the table it is seen that the spreading factor of globulin is 0.93, that of albumin 1.04 and that of total protein 1.01. In some cases the deviations from these averages are rather considerable while the cause cannot be established with certainty. In some cases HOOFT [2]) found even greater deviations for euglobulin and pseudoglobulin, however.

It cannot be said with certainty what is the cause of the very low spreading factor previously found for globulin (0.60—0.62). Undoubtedly the total protein percentage then determined by using a gravimetric method is too high. This is already seen from factor 7 (see 1) by which the nitrogen percentage found by KJELDAHL's method had to be

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[5]) B. M. KAGAN, Journ. of clinical Investigation. **17**, 369 (1938).

multiplied in order to bring it in accordance with the protein percentage found by the gravimetric method. Slight deviations from case to case of factor 6.30 which we have taken now are possible, but a value of 7 is certainly too high. The too high value of the total protein percentage gives much too high values for the globulin percentage (this was then determined as the difference between total protein and albumin percentage) and consequently the spreading factor is much too high. This cause, however, is not sufficient to bring the factor found for globulin to the value of 0.93 found now.

*Summary.*

Serum albumin and globulin were determined by means of nitrogen determinations according to the KJELDAHL method and by means of spreading. Average spreading factors of 0.93 for globulin, 1.04 for albumin and 1.01 for total protein were found.

**Physics.** — *Meson theories in five dimensions.* By L. ROSENFELD. (Communicated by Prof. H. A. KRAMERS.)

(Communicated at the meeting of January 31, 1942.)

In spite of the attractiveness of its basic idea, the meson field theory of nuclear systems cannot be said to be firmly established in any definite form. Quite apart from the convergence difficulties inherent in any quantum field theory, one is here confronted from the start with a choice between four *a priori* possible types (1) of meson fields: scalar, vector, and the two dual types with respect to spatial reflexions, pseudoscalar and pseudovector. One may then try to examine which choice provides the widest scope for the theory, including not only an account of properties of nuclear systems, but also a theory of $\beta$-disintegration, which in particular involves a definite relation between $\beta$-decay constants and the mean life time of free mesons. From this point of view, it appears necessary to adopt a particular combination of a pseudoscalar and a vector meson field, characterized by a simple relation between the constants which define the intensities of the nuclear sources of the meson fields (2) (3).

Recently, MøLLER (4) has pointed out that this "mixed theory" presents itself in a very natural way as a *single* type of meson field in a five-dimensional (pseudo-euclidian) space, *viz.* as a five-vector with respect to the group of ordinary five-dimensional "rotations" (of determinant $+1$) [1]. Moreover, such a representation of the mixed theory leads to an essential reduction of the number of arbitrary constants in the source densities of the meson field. The physical interpretation of the fifth coordinate introduces, however, an element of arbitrariness in the theory. One might, as originally proposed by MøLLER, identify the five-dimensional space with DE SITTER's universe, thus suggesting a somewhat unexpected connexion between nuclear forces and cosmological features. An alternative interpretation consists in considering the five-dimensional space as a projective one, according to VEBLEN's original suggestion (5): this has the advantage of permitting a straightforward treatment of the interaction of the mesons and nucleons with the electromagnetic field; a detailed discussion of this possibility has recently been carried out by PAIS (6).

The special position, thus recognized, of the mixed theory as a fundamental type of five-dimensional meson field raises at once the question as to which other types of such fields would also be possible *a priori*. A convenient starting point for discussing this question is provided by the so-called "particle aspect" of meson theory, *i.e.* a linearized form of the field equations, involving a system of matrices subjected to suitable commutation rules (7). In fact, the different possible types of meson fields are then immediately given by the inequivalent irreducible representations of the algebra of these matrices. Thus, in four dimensions, we have essentially [2]) two irreducible representations, of degree 5 and 10 respectively, to which correspond the scalar and the vector type of mesons, or the two dual types, according to the reflexion properties imposed on the wave function (7). Such considerations are readily extended to five dimensions (8), with the following result: there are essentially [2]) four inequivalent irreducible representations of the extended algebra, of degrees 6, 10, 10 and 15, corresponding to a five-scalar, two distinct five-pseudovector and a five-vector type of meson field respectively.

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1) This group includes in fact both the Lorentz group and the spatial reflections, provided the latter are associated with a change of sign of the fifth coordinate. More accurately, the "mixed theory" appears as some degenerate or approximate form of the five-vector theory.

2) *i.e.* apart from a trivial representation of degree 1.

In a non-projective interpretation of the five-dimensional formalism, it is found (8) that these four types of meson fields uniquely reduce to only three types of four-dimensional theories; *viz.* the five-scalar is equivalent to the four-scalar theory, both five-pseudovector types give rise to the same four-pseudovector theory, while the five-vector type is just equivalent to the mixed theory with the reduced number of source constants. In fact, in each theory suitable covariant source densities can be defined in the usual way by means of Dirac matrices. The projective interpretation, on the other hand, leads to essentially different conclusions. The discussion of this case, which has recently been worked out by PAIS (9), starts from the basic correspondence established in a well-defined way (5) between any five-projector and a set of four-tensors of all lower and equal degrees (*e.g.* a projective five-vector defines a four-vector and a four-scalar). It is, however, possible to define in a projective way the universal four-pseudoscalar $\varepsilon_{ijkl} = \pm |\det g_{mn}|^{1/2}$ and by means of this so to modify the correspondence just mentioned that any member of the set of four-tensors be replaced by its dual with respect to spatial reflections (thus, instead of a four-vector and a four-scalar, one may, from a projective five-vector, also get a pseudovector and a scalar, or a vector and a pseudoscalar, or a pseudovector and a pseudoscalar). It then follows that from the four irreducible types of projective theories for *free* mesons any one of the four-dimensional types can be derived, as well as any combination of vector or pseudovector with scalar or pseudoscalar. But the number of possibilities is greatly reduced when due account is taken of the definition of the source densities by means of Dirac matrices. If one adopts for these sources the familiar definitions, eventually modified with respect to reflection properties by multiplication with the pseudoscalar $\varepsilon_{ijkl}$, it is readily seen that in every irreducible type of projective theory all different four-dimensional possibilities obtained in the way indicated above lead just to the same physical theory [1]). So far we thus get exactly the same result as with the non-projective interpretation, *viz.* the scalar, the pseudovector and the mixed theory.

Still, the projective interpretation allows of a greater freedom in the definition of the source densities than the non-projective standpoint, because it involves a universal projector, *viz.* the coordinate vector $x^\mu$, which may be combined in a covariant way with the Dirac matrices. While this circumstance does not give rise to any essentially new possibility in the five-scalar and five-pseudovector theories, it leads for the five-vector type, in addition to the mixed theory, also to a pure vector and a pure pseudoscalar field. Summing up, we see that the five-dimensional point of view, in its widest interpretation, does not exclude any one of the four-dimensional types of meson theories, but singles out the mixed theory as the only combination of four-dimensional types which can be derived from an *irreducible* five-dimensional type of field [2]).

Whatever the formal aspect of the problem may be, the adoption of some particular form of meson theory (if any) can of course only be decided on physical arguments. If we first consider the application of meson theory to the phenomena of $\beta$-disintegration, an essential requirement in this respect is to avoid the difficulty, pointed out by NORDHEIM (11), of reconciling on such a theory the empirical value of the mean life time of the mesons with the $\beta$-decay constants of light elements. This may be achieved [3])

[1]) For the cases of the four-scalar and four-vector theories, a similar conclusion has also been reached by M. SCHÖNBERG in a recent note (10). He therefore proposes to include any pair of dual cases (scalar-pseudoscalar, vector-pseudovector) in a single type of meson theory. It would seem more practical, however, to retain the usual classification.

[2]) The reduction of the number of source constants in the mixed theory, which was stressed by MøLLER (4) as an important feature of the non-projective point of view, is not strictly implied in the projective interpretation, though it still appears as a consequence of the *simplest* choice of source densities in this case.

[3]) A quite different possibility, involving, however, the cutting-off of a divergent expression, has been pointed out by S. SAKATA, Proc. phys.-math. Soc. Japan **23**, 283 (1941).

either by adopting a purely pseudoscalar theory or by introducing two independent kinds of mesons of very different life times (3). The latter case may just be provided by the mixed theory; more precisely (3), one has here to assume, taking the five-dimensional form of the theory with the reduced number of source constants, that the pseudoscalar mesons have a much longer mean life than the vector mesons. Either one of these two possibilities thus leads to the conclusion that cosmic ray mesons observed at sea-level, being of pseudoscalar type, should have zero spin, — a conclusion strikingly supported by the analysis (12) of recent cosmic ray observations.

While such phenomena therefore appear to be in harmony with the consequences of meson theory, they do not permit to decide between pseudoscalar or mixed theory. The adoption of the latter seems to be claimed, however, for a rational treatment of nuclear forces (2). It is true that the issue in this respect is somewhat obscured by the inevitable occurrence of the well-known divergences inherent in any quantum field theory. Still, adopting a point of view analogous to the "correspondence" method of quantum electrodynamics, it is possible first to discuss the convergence of the "classical" meson theory obtained by neglecting all quantum effects of the meson field, and then to examine how the validity of such classical calculations has to be restricted in order to keep off quantum singularities. The "classical" interaction potential between a pair of nucleons at (mean) distance $r$ from each other is thus found to consist of a "static" potential and a series of non-static terms, the order of magnitude of which, in comparison with the static potential, is given by some power of the parameter $\Gamma/(\varkappa r)^n$, where $\varkappa^{-1}$ denotes the range of nuclear forces and $\Gamma \sim g^2/4\pi\hbar c \sim 0.065$ the intensity of nuclear sources of meson fields, while the exponent $n$ depends on the type of meson theory considered. On pseudoscalar as well as vector meson theory, there occurs in the static potential a dipole interaction term in $r^{-3}$, which must be cut off at some distance smaller than the range; owing to this singular term, one has in this case $n = 3$, from which it follows that the static potential in no way approximates the interaction in the region comprised between the cut-off distance and the range, where a quantitative expression for this interaction is at all of any significance. The mixed theory, on the other hand, is just defined in such a way that the singular dipole interaction term is eliminated from the static potential; one has then $n = 1$ and the inconsistency just mentioned disappears[1]). Of course, the divergences arising from the quantization of the meson field severely restrict the domain of validity of the mixed theory; the critical distance for which it breaks down, however, may, according to Heisenberg, be defined by $\Gamma/(\varkappa r_0)^2 = 1$, so that there still remains a region, between $r_0$ and $\varkappa^{-1}$, where — in contrast to pseudoscalar or vector meson theory — it yields unambiguous results.

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[1]) Explicit calculations of non-static interaction terms, which very instructively illustrate the general argument here summarized, have been published by E. STUECKELBERG and J. PATRY (13) and E. STUECKELBERG (14). As regards the numerical results given there, it must be observed that, owing to the assumption $\Gamma = 0.1$ instead of $\sim 0.065$, they perhaps convey an overpessimistic impression of the convergence of the mixed theory. The main interaction terms arising from the quantization of the meson fields have also been calculated by several authors; see especially E. STUECKELBERG and J. PATRY, *loc. cit.* (13), § 7 and H. BETHE, *loc. cit.* (15), p. 272; the calculations of MøLLER and ROSENFELD quoted by BETHE (from a verbal communication) have, however, not been published. For the vector theory, the ratio of the quantum interaction terms of order $\Gamma^2$ to the static potential is found, as mentioned by BETHE, to be of the order of magnitude $\Gamma/(\varkappa r)^2$; for the mixed theory, however, according to the unpublished calculations just referred to, this ratio becomes $\Gamma/(\varkappa r)^4$. According to the "correspondence" interpretation, all such terms have to be discarded.

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**Geophysics.** — *On the* STONELEY-*wave equation.* II. By J. G. SCHOLTE. (Communicated by Prof. J. D. V. D. WAALS.)

(Communicated at the meeting of November 29, 1941.)

§ 3. *Discussion of the* STONELEY *equation.*

In the preceding paragraph we found that the roots $\zeta$ of this equation must be less than 1; we shall now prove that these roots cannot be negative.

Putting

$$\sqrt{(1-\zeta)(1-\nu_1\zeta)}=1-\varepsilon_1\zeta, \qquad \sqrt{(1-a\zeta)(1-\omega\zeta)}=1-\omega\varepsilon_2\zeta,$$
$$\sqrt{(1-\nu_1\zeta)(1-\omega\zeta)}=1-\varphi_1\zeta, \qquad \sqrt{(1-a\zeta)(1-\zeta)}=1-\varphi_2\zeta,$$

we have $1>\varepsilon_1>\nu_1$, $1>\varepsilon_2>\nu_2$, $1>\varphi_2>\nu_2$. Equation (2) takes the form:

$$8-4\zeta\left\{\frac{1-\varrho_2/\varrho_1}{1-\mu_2/\mu_1}+\varepsilon_1+\omega\varepsilon_2\right\}+\zeta^2\left\{\left(\frac{1-\varrho_2/\varrho_1}{1-\mu_2/\mu_2}\right)^2+4\,\omega\varepsilon_1\varepsilon_2\right\}=$$
$$8-4\zeta\left\{\frac{1-\varrho_2/\varrho_1}{1-\mu_2/\mu_1}+\varepsilon_1+\omega\varepsilon_2\right\}+\zeta^2\left\{\left(\frac{1+\varrho_2/\varrho_1}{1-\mu_2/\mu_1}\right)^2+\frac{4\,\omega\varepsilon_2-4\varrho_2/\varrho_1\,\varepsilon_1}{1-\mu_2/\mu_1}\right\}-$$
$$-\zeta^3\cdot\frac{(\varrho_2/\varrho_1)^2\,\varepsilon_1+\omega\varepsilon_2+\varrho_2/\varrho_1\,\varphi_2+\varrho_2/\varrho_1\,\varphi_1}{(1-\mu_2/\mu_1)^2}$$

or

$$\zeta\left\{\frac{\mu_1}{\mu_1}\varepsilon_2+\frac{\varrho_2}{\varrho_1}\varepsilon_1+\varphi_1+\varphi_2\right\}=4\left\{1-\varepsilon_1\left(1-\frac{\mu_2}{\mu_1}\right)+\right.$$
$$\left.+\varepsilon_2\left(\frac{\mu_1}{\mu_2}-1\right)-\varepsilon_1\varepsilon_2\frac{\mu_1}{\mu_2}\left(1-\frac{\mu_2}{\mu_1}\right)^2\right\}$$

hence $\zeta=\dfrac{4\,\{1-\varepsilon_1(1-\mu_2/\mu_1)\}\,\{1-\varepsilon_2(1-\mu_1/\mu_2)\}}{\varrho_2/\varrho_1\,\varepsilon_1+\mu_1/\mu_2\,\varepsilon_2+\varphi_1+\varphi_2}$, which is positive, $\varepsilon_1$ and $\varepsilon_2$ being less than 1.

As we have now proved that $0<\zeta<1$, it follows that *sin* $r_1$ is real and greater than $1\left(\zeta=\dfrac{1}{\sin^2 r_1}\right)$, hence the cosines of the angles occurring in equation (1) are imaginary, and the wave function

$F(pt-h_1\,x\sin i_1-h_1\,z\cos i_1)$ becomes $F(pt-h_1\,x\sin i_1-i\,h_1\,z\sqrt{\sin^2 i_1-1})$.

The waves of the system $\{A_r\,\mathfrak{A}_r\,A_d\,\mathfrak{A}_d\}$ are therefore exponentially damped in the $z$ direction.

It is convenient once more to choose a new variable, namely $\eta=\dfrac{1}{\zeta}=\sin^2 r_1$. Equation (2) is then:

$$\left.\begin{aligned}&\left(2\eta-\frac{1-\varrho_2/\varrho_1}{1-\mu_2/\mu_1}\right)^2+4\sqrt{(\eta-1)(\eta-\nu_1)(\eta-a)(\eta-\omega)}=\\&\sqrt{\frac{(\eta-a)(\eta-\omega)}{\eta^2}}\cdot\left(2\eta-\frac{1}{1-\mu_2/\mu_1}\right)^2+\sqrt{\frac{(\eta-\nu_1)(\eta-1)}{\eta^2}}\cdot\left(2\eta+\frac{\varrho_2/\varrho_1}{1-\mu_2/\mu_1}\right)^2+\\&+\frac{\varrho_2/\varrho_1}{(1-\mu_2/\mu_1)^2}\cdot\left(\sqrt{\frac{(\eta-a)(\eta-1)}{\eta^2}}+\sqrt{\frac{(\eta-\nu_1)(\eta-\omega)}{\eta^2}}\right)\end{aligned}\right\}\quad(3)$$

while the roots $\eta$ of this equation are $> 1$.

We are now prepared to solve the major problem concerning the STONELEY waves, namely: what is the condition for the two media that must be fulfilled if STONELEY waves shall be possible at their interface?

First we shall investigate whether equation (3) can be satisfied by very large values of $\eta$; if $\eta$ is very large the value of the left-hand side $(L)$ is approximately:

$$L \approx 8\eta^2 - 4\eta \cdot \left(\frac{1-\varrho_2/\varrho_1}{1-\mu_2/\mu_1} + \frac{1+\nu_1+\omega+\omega\nu_2}{2}\right) +$$
$$+ \left(\frac{1-\varrho_2/\varrho_1}{1-\mu_2/\mu_1}\right)^2 + \left(\nu_1 + a + \omega + \nu_1 a + \nu_1 \omega + a\omega - \frac{1+\nu_1^2+a^2+\omega^2}{2}\right).$$

The value of the right-hand side $(R)$ proves then to be

$$A \approx 8\eta^2 - 4\eta \cdot \left(\frac{1-\varrho_2/\varrho_1}{1-\mu_2/\mu_1} + \frac{1+\nu_1+\omega+\omega\nu_2}{2}\right) + \left(\frac{1+\varrho_2/\varrho_1}{1-\mu_2/\mu_1}\right)^2 +$$
$$+ \left(\nu_1 + a\omega - \frac{1+\nu_1^2+a^2+\omega^2}{2}\right) + \frac{2}{1-\mu_2/\mu_1} \cdot \left\{(\omega+\nu_2\omega) - \frac{\varrho_2}{\varrho_1}(1+\nu_1)\right\}.$$

Hence

$$L\text{–}R \approx (\nu_1 a + \nu_1 \omega + a + \omega) - \frac{2}{1-\mu_2/\mu_1} \cdot \{\omega(1+\nu_2) - \varrho_2/\varrho_1 (1+\nu_1)\} - 4\frac{\varrho_2/\varrho_1}{(1-\mu_2/\mu_1)^2}$$

or

$$\approx \left\{(1+\nu_1) - \frac{2}{1-\mu_2/\mu_1}\right\}\left\{\omega(1+\nu_2) + \frac{2\varrho_2/\varrho_1}{1-\mu_2/\mu_1}\right\}.$$

This difference is equal to zero when $\frac{\mu_2}{\mu_1} = -\frac{1-\nu_1}{1+\nu_1}$ and $\frac{\mu_2}{\mu_1} = -\frac{1-\nu_2}{1+\nu_2}$; these values are both negative, as $\nu = \frac{1}{2+\lambda/\mu} < \frac{1}{2}$.

As $L - R < 0$ if $\mu_2/\mu_1 = 0$, it follows that $L - R < 0$ if $\mu_2/\mu_1$ is greater than the largest of the values $-\frac{1-\nu_1}{1+\nu_1}$ and $-\frac{1-\nu_2}{1+\nu_2}$.

The left-hand side $L$ of equation (3) is consequently for large values of $\eta$ smaller than the right-hand side $R$ for every value of $\mu_2/\mu_1$ which is physically possible.

We arrive therefore at the conclusion that very large roots $\eta$ are impossible.

Our knowledge concerning equation (3) now amounts to the following:

1. $L < R$ if $\eta$ is very large.
2. $L = R$ is only possible if $\eta > 1$,

therefore we can definitely assert that equation (3) has a root if $L > R$ for $\eta = 1$.

We are thus induced to the investigation of the value of $L - R$ if $\eta = 1$.

Substituting $\eta = 1$ we get:

$$L = \left(2 - \frac{1-\varrho_2/\varrho_1}{1\ \ \mu_2/\mu_1}\right)^2 = \frac{(1-2\mu_2/\mu_1)^2 + \varrho_2/\varrho_1\{\varrho_2/\varrho_1 + 2(1-2\mu_2/\mu_1)\}}{(1-\mu_2/\mu_1)^2}$$

$$R = \frac{\sqrt{(1-\nu_2\omega)(1-\omega)} \cdot (1-2\mu_2/\mu_1) + \varrho_2/\varrho_1 \cdot \sqrt{(1-\nu_1\omega)(1-\omega)}}{(1-\mu_2/\mu_1)^2} =$$

$$= \frac{(1-2\mu_2/\mu_1)^2(1-\varepsilon_2\omega) + \varrho_2/\varrho_1(1-\varepsilon_1)}{(1-\mu_2/\mu_1)^2}$$

$$= \frac{(1-2\mu_2/\mu_1)^2 + \varrho_2/\varrho_1\{(1-\varepsilon_1) - \mu_1/\mu_7\,\varepsilon_2(1-2\mu_2/\mu_1)^2\}}{(1-\mu_2/\mu_1)^2}$$

hence $L - R > 0$ if

$$\frac{\varrho_2}{\varrho_1} + 2\left(1 - 2\frac{\mu_2}{\mu_1}\right) > (1-\varepsilon_1) - \frac{\mu_1}{\mu_2}\varepsilon_2\left(1 - 2\frac{\mu_2}{\mu_1}\right)^2$$

or

$$2 - \frac{\mu_2}{\mu_1}(4-\omega) > -\frac{\mu_1}{\mu_2}\varepsilon_2\left(1 - 4\frac{\mu_2}{\mu_1} + 4\frac{\mu_2^2}{\mu_1^2}\right) + 1 - \varepsilon_1$$

or

$$\left(\frac{\mu_2}{\mu_1}\right)^2 . (4 - 4\varepsilon_2 - \omega) + \left(\frac{\mu_2}{\mu_1}\right) . (4\varepsilon_2 - \varepsilon_1 - 1) - \varepsilon_2 < 0 \quad . \quad . \quad . \quad (4)$$

It follows that the STONELEY equation has a root if this quadratic inequality is satisfied.

The discussion of this inequality is very simple; taking first the discriminant $D$ of the quadratic we get:

$$\begin{aligned} D &= (4\varepsilon_2 - \varepsilon_1 - 1)^2 + 4\varepsilon_2(4 - 4\varepsilon_2 - \omega) \\ &= (1+\varepsilon_1)^2 - 4\varepsilon_2(\omega + 2\varepsilon_1 - 2) \end{aligned}$$

and this is negative if

$$\varepsilon_2 > \frac{(1+\varepsilon_1)^2}{4(\omega + 2\varepsilon_1 - 2)}$$

hence

$$1 > \frac{(1+\varepsilon_1)^2}{4(\omega + 2\varepsilon_1 - 2)}, \text{ as } \varepsilon_2 < 1;$$

or, after some reduction: $3 - \varepsilon_1 < 2\sqrt{\omega}$, which is obviously impossible as $\varepsilon_1 < 1$ and $\omega < 1$.

The discriminant being therefore always positive, the quadratic function (4) is always equal to zero for two real values of $\mu_2/\mu_1$.

It may be remarked that this calculation is only valid if $\omega < 1$, that is: $\mathfrak{B}_2 > \mathfrak{B}_1$. If $\mathfrak{B}_2 = \mathfrak{B}_1$, $(\omega = 1)$ the quadratic function reduces to $-\left(\frac{\mu_2}{\mu_1}\right)^2 + 2\frac{\mu_2}{\mu_1} - 1$ which is, of course, always negative. The two roots of the quadratic equation

$$\left(\frac{\mu_2}{\mu_1}\right)^2 (4 - 4\varepsilon_2 - \omega) + \left(\frac{\mu_2}{\mu_1}\right)(4\varepsilon_2 - \varepsilon_1 - 1) - \varepsilon_2 = 0 \quad . \quad . \quad . \quad (5)$$

coincide here at the value $\mu_2/\mu_1 = 1$.

Continuing with the discussion of the quadratic function (4), the coefficient of $\mu_2/\mu_1$ must now be investigated. This factor is negative if $1 + \varepsilon_1 > 4\varepsilon_2$; as

$$\varepsilon_1 = 1 - \sqrt{(1-\nu_1)(1-\omega)} < 1 - \sqrt{{}^1/_2(1-\omega)}, \ \nu_1 \text{ being } = \frac{1}{2 + \lambda_1/\mu_1} < {}^1/_2, \text{ and}$$

$$\omega\,\varepsilon_2 = 1 - \sqrt{(1-\nu_2\,\omega)(1-\omega)} \geqq 1 - \sqrt{1-\omega}, \ \nu_2 \text{ being } = \frac{1}{2 + \lambda_2/\mu_2} \geqq 0$$

it is evident that if the inequality $1 + \varepsilon_1 > 4\varepsilon_2$ is true, then *à fortiori*

$$2 - \sqrt{{}^1/_2(1-\omega)} > 4 . \frac{1 - \sqrt{1-\omega}}{\omega}$$

or

$$4-2\,\omega < (4-{}^1/_2\,\omega\sqrt{2})\sqrt{1-\omega}\,;$$

squaring and reducing this inequality gives: $\omega^2+\omega(7-8\sqrt{2})+8\sqrt{2}<0$, which is false.

The coefficient of $\mu_2/\mu_1$ too is always positive, from which it is shown that the signs of the roots of equation (5) are entirely dependent on the coefficient of $(\mu_2/\mu_1)^2$.

This coefficient

$$= 4-4\,\varepsilon_2-\omega$$

$$= 4-\omega-4\,.\,\frac{1-\sqrt{1-\nu_2\,\omega)\,(1-\omega)}}{\omega}$$

$$= \frac{\sqrt{(1-\nu_2\,\omega)\,(1-\omega)}-(1-{}^1/_2\,\omega)^2}{\omega}.$$

The equation $\sqrt{(1-\nu_2\,\omega)\,(1-\omega)}-(1-\tfrac{1}{2}\omega)^2=0$ has, for every value of $\nu_2(<\tfrac{1}{2})$, one real root as can be easily proved as follows: Writing $\dfrac{S_2^2}{\mathfrak{V}_2^2}$ for $\omega$ this equation becomes

$$\sqrt{\left(1-\frac{S_2^2}{V_2^2}\right)\left(1-\frac{S_2^2}{\mathfrak{V}_2^2}\right)}=\left(1-\frac{1}{2}\frac{S_2^2}{\mathfrak{V}_2^2}\right)^2$$

and this is the velocity equation for simple RAYLEIGH waves which are propagated in the second medium with the velocity $S_2$. This equation has always one real root $S_2 < \mathfrak{V}_2$ (LAMB [3]).

The coefficient of $(\mu_2/\mu_1)^2$ is therefore equal to zero if $\dfrac{S_2^2}{\mathfrak{V}_2^2}=\omega$ or if $\mathfrak{V}_1=S_2$, as $\omega=\dfrac{\mathfrak{V}_1^2}{\mathfrak{V}_2^2}$.

For very small values of $\omega$ this coefficient is approximately

$\approx\left(1-\dfrac{1+\nu_2}{2}\,\omega\right)-(1-\omega)=\tfrac{1}{2}\,\omega\,(1-\nu_2)$, which is positive.

The coefficient of $(\mu_2/\mu_1)^2$ of the quadratic (4) is consequently positive if $\mathfrak{V}_1<S_2$, equal to zero if $\mathfrak{V}_1=S_2$, and negative if $\mathfrak{V}_1>S_2$.

Combining this result with the already derived properties of the quadratic form (4) we find that the roots of equation (5) are: both positive if $\mathfrak{V}_1>S_2$; one positive and the other infinite if $\mathfrak{V}_1=S_2$; one positive and the other negative if $\mathfrak{V}_1<S_2$; both equal to 1 if $\mathfrak{V}_1=\mathfrak{V}_2$.

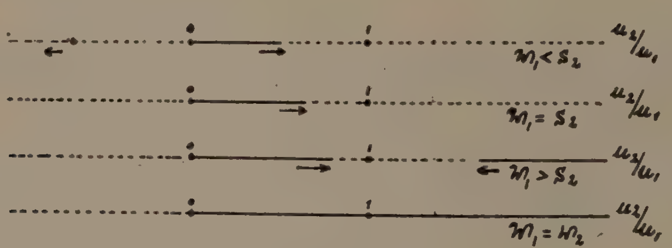


Fig. 3.

The general course of the roots $\mu_2/\mu_1$ of equation (5) can be seen in figure 3. One root is always positive and increases in value if $\mathfrak{V}_1/\mathfrak{V}_2$ diminishes; in this case the other root always decreases, being negative if $\mathfrak{V}_1<S_2$, infinite if $\mathfrak{V}_1=S_2$ and positive if

$\mathfrak{V}_1 > S_2$; both roots approach the same value $\mu_2/\mu_1 = 1$ if $\mathfrak{V}_2 \approx \mathfrak{V}_1$, which value is reached if $\mathfrak{V}_2 = \mathfrak{V}_1$. Those positive values of $\mu_2/\mu_1$, for which inequality (4) is satisfied, are indicated by the heavy lined parts of the $\mu_2/\mu_1$ axis.

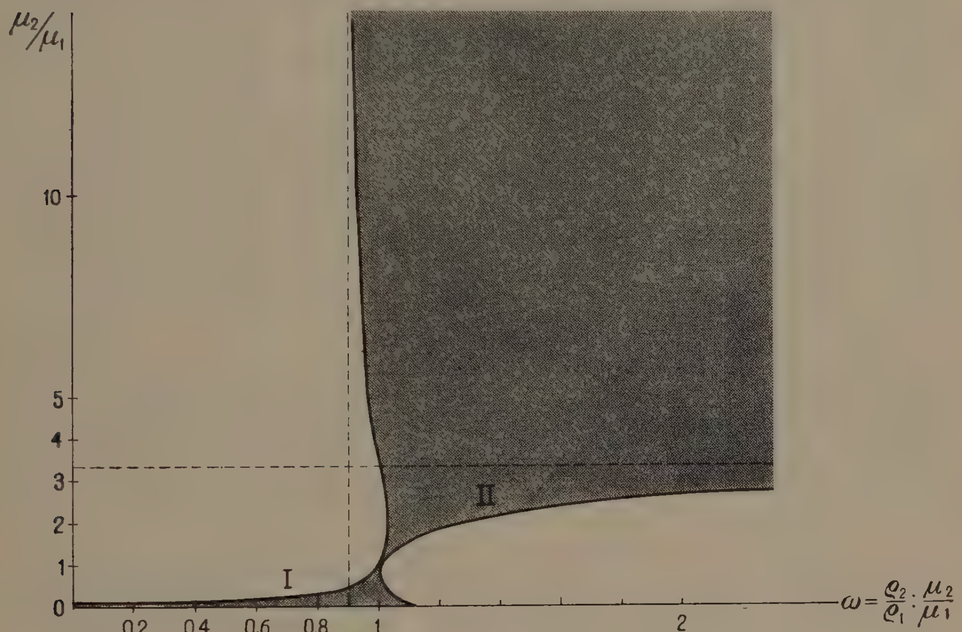


Fig. 4.

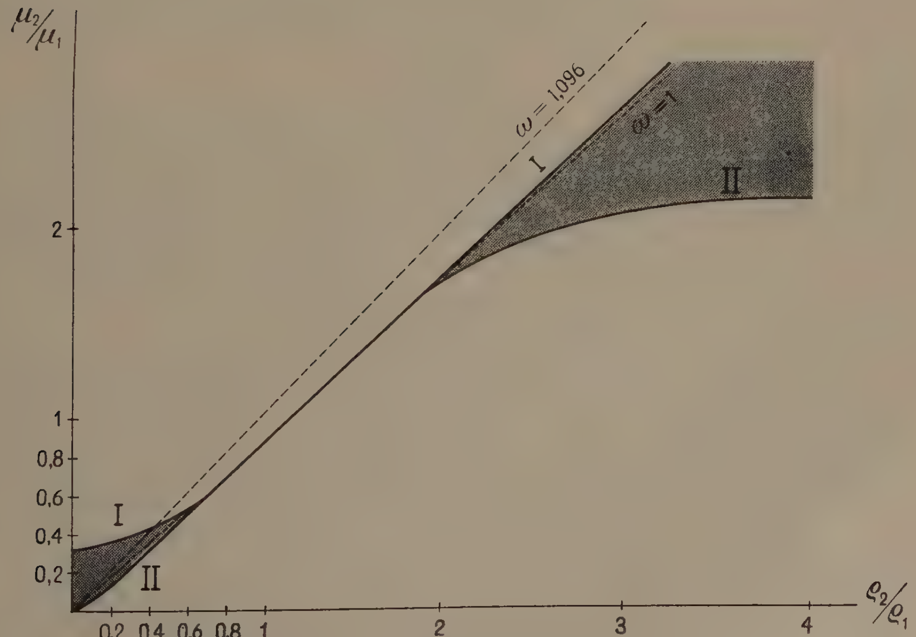


Fig. 5.

Remembering that the STONELEY equation has a root if inequality (4) holds good, it is at once evident that the just mentioned intervals are also the intervals where a STONELEY wave system is possible. The areas where STONELEY waves are possible are shown more completely in diagrams 4 and 5. In the calculations use is made of the fact that equation (1) is symmetrical with respect to the suffixes 1 and 2, so that the case $\mathfrak{V}_1 > \mathfrak{V}_2$ can be calculated by changing these suffixes.

It will be obvious that equation (5) is the equation of curve I ($\mathfrak{V}_1 < \mathfrak{V}_2$), which can be reduced to

$$\left(1-2\frac{\mu_2}{\mu_1}+\frac{\varrho_2}{\varrho_1}\right)^2=\left\{\left(1-2\frac{\mu_2}{\mu_1}\right)^2\sqrt{1-\nu_2}+\frac{\varrho_2}{\varrho_1}\sqrt{1-\nu_1}\right\}\sqrt{1-\frac{\varrho_2/\varrho_1}{\mu_2/\mu_1}};$$

the equation of curve II ($\mathfrak{V}_1 < \mathfrak{V}_{12}$) is therefore

$$\left(1-2\frac{\mu_1}{\mu_2}+\frac{\varrho_1}{\varrho_2}\right)^2=\left\{\left(1-2\frac{\mu_1}{\mu_2}\right)^2\sqrt{1-\nu_1}+\frac{\varrho_1}{\varrho_2}\sqrt{1-\nu_2}\right\}\sqrt{1-\frac{\varrho_1/\varrho_2}{\mu_1/\mu_2}}.$$

Both figures relate to the case of incompressible media ($\lambda = \infty$).

As a comparison is now possible between our results and those of LOVE [4]) and SEZAWA [5]) and as the shape of the curves is not essentially altered by taking other values of $\lambda$, I have chosen the $\lambda$ of the case under discussion to execute the comparison, the results of which will be communicated in a following paper.

## § 4. *Summary.*

The STONELEY wave system can be derived by an extension of the calculations of KNOTT, relating to the reflection of elastic waves at the surface of separation between two infinite media. The corresponding wave equation is not always solvable, as has been pointed out by STONELEY. In the present paper it is shown for what values of the material constants ($\varrho_2/\varrho_1$ and $\mu_2/\mu_1$) the equation can be solved.

I wish to express my thanks to Prof. J. D. V. D. WAALS for his kind interest in this paper.

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**Mathematics.** — *Sur quelques inégalités de la théorie des fonctions et leurs généralisations spatiales.* II. Par A. F. MONNA. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of December 27, 1941.)

§ 4. *Généralisations spatiales.*

1. Au § 2 nous avons déjà vu qu'une généralisation spatiale du théorème de KOEBE n'est possible que dans des cas particuliers. Dans toute sa généralité le théorème modifié n'est vrai que dans l'espace deux-dimensionnel. Ces cas particuliers se rattachent au théorème, donné au fin du § 2. En tenant la notation de ce théorème, on a

$$g\,(P_0) \geqq g^*\,(P_0) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (21)$$

où l'on a supposé que le point $P_0$ se trouve intérieur à $\Omega^*$ et aux $\Omega$. Considérons dans $\Omega$ la différence $g(P, P_0) - g^*(P, P_0)$. Puisque $g(P, P_0) < \frac{1}{r_{P,P_0}}$, on voit que cette diffférence, fonction harmonique de $P$, correspond à des valeurs-frontière positive sur $\Sigma$ de sorte que, en vertu du principe de maximum, la différence est positive pour tout $P$ dans $\Omega$. En particulier on a l'inégalité ci-dessus.

Cette généralisation est le maximum qu'on peut atteindre. Cela veut dire, si l'on considère tous les $\Omega^*$, la plus grande borne inférieure de l'ensemble des valeurs $g^*(P_0)$ est égale à zéro. Pour le voir il suffit de prendre pour $\Sigma^*$ un cône circulaire droite de sommet $O$ et pour $\Omega^*$ la partie de l'espace extérieur à ce cône; $g^*(P_0)$ est alors positif. Si l'on fait tendre vers zéro l'angle solide du cône, $\Omega^*$ tend vers le „Schlitzgebiet" trois-dimensionnel et pour ce domaine $g(P_0)$ est zéro, même $g(P, P_0) \equiv 0$ puisque la frontière est de capacité nulle. On ne peut donc améliorer (21) qu'a

$$g\,(P_0) \geqq 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (22)$$

et cela est trivial puisque les valeurs-frontière sont positives.

Une formule analogue à (14) subsiste dans ces Cas particuliers. Considérons de nouveau les domaines $\Omega$ intérieurs à $\Omega^*$ dont la frontière $\Sigma$ passe par $O$ et qui contiennent tous le point $P_0$ de $\Omega^*$. Supposons de plus que la distance $d$ de $P_0$ à $O$ est la plus petite distance de $P_0$ à $\Sigma$. En appliquant la transformation (17) on démontre alors tout analogue, de sorte que nous omettons l'épreuve [1]),

$$g\,(P_0) \geqq \frac{k\,(\Omega^*)}{d} \qquad (k > 0) \quad . \; . \; . \; . \; . \; . \quad (23)$$

Par $k\,(\Omega^*)$ nous indiquons que $k$ dépend de $\Omega^*$. Comme on a vu ci-dessus (inégalité (22)) la plus grande borne inférieure de $k(\Omega^*)$ vaut zéro ($d$ constant, $P_0$ fixe). Ceci en contradiction avec le cas deux-dimensionnel. Le théorème, exprimé par (21) subsiste naturellement en ce dernier cas, mais alors la borne inférieure de $k$ est différent de la borne triviale, qui est $-\infty$ dans ce cas; c'est justement le théorème de KOEBE.

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[1]) La formule (23) subsiste si la distance de $P_0$ à $\Sigma$ est $< d$, toujours en supposant que $P_0$ est intérieur à $\Omega$ et $O$ est un point de $\Sigma$. Mais alors la borne (23) peut être amélioriée. Le même se passe pour les formules suivantes (25) et (29).

En considérant les domaines pour lesquels $g(P_0) = 1$, on montre comme dans le cas deux-dimensionnel l'existence des fonctions $\varphi(C, \Omega^*)$ et $\psi(C, \Omega^*)$, valable dans les cas particuliers mentionnés. Le passage au cas $g(P_0) \neq 1$ est maintenant un peu différent. C'est puisque la surface $G = C$ est maintenant transformée dans la surface $\overline{G} = \frac{C}{p}$. On trouve alors les rayons

$$\varphi\left(\frac{C}{g\,(P_0)};\ \Omega^*\right)\frac{1}{g\,(P_0)} \quad \text{et} \quad \psi\left(\frac{C}{g\,(P_0)};\ \Omega^*\right)\frac{1}{g\,(P_0)}. \quad . \quad . \quad (24)$$

Remarquons enfin que pour la validité de (21) il n'est pas nécessaire, ni dans le cas deux-dimensionnel ni dans le cas trois-dimensionnel, que $\Omega^*$ et $\Omega$ sont simplement connexes. Cependant si l'on admet que $\Omega^*$ puisse être multiplement connexe, la borne inférieure de $k$ vaut $-\infty$ dans le cas deux-dimensionnel, de sorte qu'alors le théorème de KOEBE ne subsiste plus.

2. Traitons enfin le cas spécial suivant.

Prenons pour $\Omega^*$ le demi-espace à droite d'un plan $V$ et soit $O$ un point dans $V$. Considérons les domaines $\Omega$ intérieurs à $\Omega^*$, dont la frontière passe par $O$ et qui contiennent tous le point $P_0$ de $\Omega^*$; $P_0\,O$ est supposé perpendiculaire à $V$. On a alors

$$g\,(P_0) \geqq \frac{1}{2\,d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

si $d$ désigne la distance $P_0\,O$, la plus petite distance de $P_0$ à $\Sigma$ [1]).

Nous savons par ce qui précède que le minimum de $g(P_0)$ est atteint pour $\Omega^*$ et il suffit donc de déterminer $g^*(P_0)$. On a

$$g^*\,(P_0) = \int\limits_{r} \frac{1}{r_{P_0 Q}}\, d\,\mu^{P_0}\,(Q)$$

où $\mu^{P_0}(e)$ désigne la mesure harmonique de e par rapport à $\Omega^*$. Cette mesure est connue (voir la démonstration de l'inégalité (6)). On trouve alors

$$g^*\,(P_0) = \int\limits_0^\infty \frac{1}{\sqrt{d^2 + R^2}}\, d_R \left(1 - \frac{d}{\sqrt{d^2 + R^2}}\right)$$

$$= d \int\limits_0^{\frac{1}{d}} \varrho\, d\varrho = \frac{1}{2\,d}$$

ce qu'il fallait démontrer.

Si $g(P_0) = 1$, on a $d \geqq \frac{1}{2}$. En appliquant ceci, on trouve dans notre cas

$$\varphi\,(C, \Omega^*) \geqq \frac{1}{C + 2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Les formules (25) et (26) expriment une extension du théorème suivant de la théorie des fonctions:

Soit

$$w = f\,(z) = a_1\, z + a_2\, z^2 + \ldots.$$

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1) Voir note 1) pag. 165.

régulier et „schlicht" dans le cercle unité et supposons de plus que ce cercle est représenté par $f(z)$ sur un domaine convexe $\Omega$.

On a alors

$$\frac{|z|}{1+|z|}\,|a_1| \leqq |f(z)| \leqq \frac{|z|}{1-|z|}\,|a_1|. \quad . \quad . \quad . \quad . \quad (27)$$

Exprimé autrement:

la courbe $G = C$, $G$ étant la fonction de GREEN de $\Omega$, se trouve entre les deux cercles de centre $P_0$ et de rayon respectivement

$$\frac{e^{-c}}{1+e^{-c}}\, e^{-g(P_0)} \quad \text{et} \quad \frac{e^{-c}}{1-e^{-c}}\, e^{-g(P_0)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

En particulier on trouve pour $C = 0$ ($d$ distance de $P_0$ à $\Sigma$)

$$g(P_0) \geqq \log \frac{1}{2d}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

La fonction $\frac{z}{1-z}$ atteint les bornes (27) et $\Omega$ est alors un demi-plan.

Les formules (25) et (29) sont donc analogues. Le même se passe pour (26) et le premier rayon de (28). Dans le cas trois-dimensionnel on trouve de (24) et (26) que la sphère de rayon

$$\frac{1}{\frac{C}{g(P_0)}+2}\;\frac{1}{g(P_0)} = \frac{1}{C+2g(P_0)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (26')$$

et de centre $P_0$ est intérieur à la surface $G = C$. Dans le cas deux-dimensionnel c'est le cercle de rayon

$$\frac{e^{-c}}{1+e^{-c}}\, e^{-g(P_0)}.$$

Les bornes (25) et (29) sont atteintes pour un demi-espace respectivement un demi-plan.

Le théorème (27) sur les fonctions convexes admet donc une généralisation et on avait pu prévoir cela par le théorème général du no. 1 de ce paragraphe, puisqu'un demi-espace est le plus grand domaine convexe qui contient les domaines convexes considérés et la capacité de la frontière de ce domaine est positive.

Toutefois il y a une petite différence. Les inégalités (25), (26) et (26′) sont valables aussi si $\Omega$ n'est pas convexe: en effet ceci n'est pas utilisé dans la démonstration. L'inégalité (29) reste vraie aussi dans ce cas, les domaines étant toujours intérieurs à un demi-plan. Mais cela n'est pas certain pour les rayons (28). Par ce qui précède on sait que des cercles, de rayon différent respectivement de 0 et $\infty$, existent mais les valeurs exactes de ces rayons ne sont pas connues. En remarquant que $d \geqq \frac{1}{2}$ si $g(P_0) = 0$, on trouve par un raisonnement analogue à celui pour obtenir (20) la borne $\frac{1}{2}e^{-C-g(P_0)}$. Cette valeur suggère que peut-être les rayons (28) sont valables encore dans le cas général de $\Omega$ non convexe intérieur à un demi-plan.

Revenons encore aux formules (25) et (26). La borne (25) ne peut pas être améliorée puisqu'elle est atteinte pour le demi-espace $\Omega^*$. La question se pose si (26) et donc (26′) expriment aussi une borne exacte. C'est une question non résolue. C'est probable que les bornes exactes soient atteintes aussi pour le demi-espace. Si cela était vrai, $\frac{1}{C+2}$ n'était pas exacte. La fonction de GREEN est explicitement connue dans ce cas. Pour la

déterminer on peut appliquer la théorie des images électriques. Soit donc $P'_0$ l'image réflétée de $P_0$ par rapport au plan $V$. En désignant par $r$ la distance de $P$ à $P_0$ et par $r'$ la distance $P P'_0$, on a

$$G(P, P_0) = \frac{1}{r} - \frac{1}{r'},$$

$$g(P, P_0) = \frac{1}{r'}.$$

Remarquons qu'on retrouve immédiatement la valeur $\frac{1}{2d}$ de l'inégalité (25). Pour déterminer le rayon de la plus grande sphère de centre $P_0$ qui est intérieure à la surface $G = C$, il faut déterminer le minimum de $r$ sous la condition $\frac{1}{r} - \frac{1}{r'} = C$. C'est donc un calcul élémentaire. On trouve que le minimum est atteint sur la droite $OP_0$ et a la valeur

$$d \frac{1 + Cd - \sqrt{1 + C^2 d^2}}{Cd}$$

Puisque $g(P_0) = \frac{1}{2d}$ dans notre cas, on voit que cette valeur est bien de la forme (24).

Au lieu de $\frac{1}{C+2}$, on trouve, en posant $d = \frac{1}{2}$, la borne

$$\frac{C + 2 - \sqrt{C^2 + 4}}{2C}$$

qui est en effet plus grande que $\frac{1}{C+2}$.

Par la même voie on trouve pour $\Psi(C)$ la valeur

$$\frac{1}{2}\left[\sqrt{1 + \frac{4}{C}} - 1\right]$$

Bien entendu, il faudrait montrer encore que les bornes exactes sont atteintes pour le demi-espace.

*Remarque.*

On peut déterminer la valeur $\log \frac{1}{2d}$ de $g(P_0)$ pour le demi-plan par une voie analogue à celle par laquelle nous avons trouvé la valeur $\frac{1}{2d}$ de l'inégalité (25). Cela donne la valeur d'une intégrale intéressante. On trouve

$$\frac{2}{\pi}\int_0^\infty \log \frac{1}{\varrho}\, d_R \operatorname{arc\,tg} \frac{R}{d} = -\frac{d}{\pi}\int_0^\infty \frac{\log(R^2 + d^2)}{R^2 + d^2}\, dR.$$

Puisque nous savons que la valeur est $\log \frac{1}{2d}$, on trouve

$$\int_0^\infty \frac{\log(R^2 + d^2)}{R^2 + d^2}\, dR = \frac{\pi}{d} \log 2d.$$

*Dordrecht*, novembre 1941.

**Mathematics.** — *La représentation conforme au voisinage d'un point frontière.* Par Prof. J. WOLFF. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of January 31, 1942.)

Mlle JACQUELINE FERRAND a démontré le théorème suivant:

*Si un domaine $\triangle$ simplement connexe du plan de la variable complexe $\zeta$ contient un secteur angulaire dont le sommet $\alpha$ est sur sa frontière, alors dans toute représentation conforme de $\triangle$ sur le demi-plan $D\,(x>0)$ du plan de la variable complexe $z=x+iy$, telle qu'au point $z=\infty$ correspond le bout premier* (Primende, *au sens de* M. C. Carathéodory) *contenant* $\alpha$,

$$\alpha = \textit{limite angulaire de } \zeta \textit{ pour } z \to \infty\,.$$ [1])

Nous montrerons que la présence du secteur angulaire de sommet $\alpha$ est superflue. En remplaçant pour plus de commodité $z=\infty$ par $z=0$ nous démontrerons donc le

**THÉORÈME. Soit $\alpha$ un point frontière accessible d'un domaine $\triangle$ du plan de la variable complexe $\zeta$, qu'on représente conformément sur le demi-plan D $(x>0)$ du plan de la variable complexe $z=x+iy$ au moyen d'une fonction $\zeta=\zeta(z)$ tel que $z=0$ correspond au bout premier qui contient $\alpha$. Alors $\alpha$ est la limite angulaire de $\zeta(z)$ pour $z\to 0$.**

**Démonstration.** Sans nuire à la généralité supposons $\triangle$ borné. Le point $\alpha$ étant accessible, $\triangle$ contient une courbe continue $\Gamma$ aboutissant en $\alpha$, image d'une courbe continue $C$ dans $D$ aboutissant en $O$ $(z=0)$. Parceque $\triangle$ est borné nous savons que, en posant $|z|=\varrho$, $arg\ z=\varphi$,

$$\int_0^\infty d\varrho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left|\frac{d\zeta}{dz}\right|^2 . \varrho\, d\varphi < \infty\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Indiquons par $L(\varrho)$, $0<\varrho<\infty$ la longueur de l'image dans $\triangle$ du demi-cercle $|z|=\varrho, -\frac{\pi}{2}<\varphi<\frac{\pi}{2}$ de $D$. Alors

$$\{L(\varrho)\}^2 = \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left|\frac{d\zeta}{dz}\right| \varrho\, d\varphi\right)^2 \leqq \pi\varrho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left|\frac{d\zeta}{dz}\right|^2 \varrho\, d\varphi\,. \quad . \quad . \quad (2)$$

De (1) et (2) on tire:

$$\int_0^\infty \{L(\varrho)\}^2 \frac{d\varrho}{\varrho} < \infty\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

L'inégalité (3) permet de conclure que, $\varrho$ tendant vers zéro, $L(\varrho)$ tend *approximativement* vers zéro, c'est à dire: quelque soit le nombre positif $\varepsilon$, la mesure $\mu(r)$, $r>0$, de l'ensemble des valeurs de $\varrho$ entre 0 et $r$ pour lesquelles $L(\varrho)>\varepsilon$ satisfait à

$$\lim_{r\to 0} \frac{\mu(r)}{r} = 0.$$

[1]) Comptes rendus de l'Ac. des Sc. de Paris, 9 Juin 1941.

Traçons dans $D$ deux demi-droites $OA$ et $OB$ différentes, du reste arbitraires, issues de $O$ $(z=0)$. Soit $\varrho$ un nombre positif. Le demi-cercle dans $D$ de centre $O$ et de rayon $\varrho$ coupe $OA$ et $OB$, soit en $z_1$ et $z_2$ respectivement, et, pourvu que $\varrho$ soit suffisamment petit, il coupe la courbe $C$, soit en $z^*$. Il est évident que

$$|\zeta(z_1)-\zeta(z^*)| \leqq L(\varrho) \text{ et } |\zeta(z_2)-\zeta(z^*)| \leqq L(\varrho). \quad . \quad . \quad . \quad . \qquad (4)$$

Or, $\varrho$ tendant vers zéro, $\zeta(z^*)$ tend vers $\alpha$ et $L(\varrho)$ tend approximativement vers zéro. Donc, en vertu de (4), sur $OA$ et $OB$ la fonction $\zeta(z)$ tend approximativement vers $\alpha$ pour $z \to 0$. Et, parceque $\zeta(z)$ est bornée dans $D$, cela entraîne que sur les demi-droites issues de O intérieures à un angle $A'OB'$ quelconque plus petit que l'angle $AOB < \pi$, et ayant même bissectrice que celui-ci, la fonction $\zeta(z)$ tend *uniformément* vers $\alpha$ pour $z \to 0$. On le voit en représentant l'angle $AOB < \pi$ conformément sur un cercle $K$ et en appliquant l'intégrale de POISSON à la transformée de la fonction $\zeta(z)$: si $w(z)$ est la fonction représentatrice, la limite approximative $\zeta \to \alpha$ pour $z \to 0$ sur $OA$ et $OB$ devient la limite approximative $\zeta \to \alpha$ sur la circonférence de $K$ pour $w \to w(o)$, des deux côtés de $w(o)$. La fonction $\zeta$ étant bornée dans $K$, l'intégrale de POISSON montre que $\alpha$ est la limite angulaire de $\zeta$ pour $w \to w(o)$ dans $K$. Par suite $\zeta$ tend vers $\alpha$ pour $z \to 0$ dans l'angle $A'OB'$.

Les demi-droites $OA$ et $OB$ étant arbitraires, le théorème est démontré.

**Mathematics.** — *Die Berechnung der vollständigen elliptischen Integrale erster und zweiter Art für grosse Werte von* $|k|$. Von S. C. VAN VEEN. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of January 31, 1942.)

Die vollständigen elliptischen Integrale erster und zweiter Art

$$K(k)=\int_0^1 \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}=\tfrac{1}{2}\int_0^1 \frac{dv}{\sqrt{v(1-v)(1-k^2v)}} \quad . \quad . \quad (1)$$

und

$$E(k)=\int_0^1 \sqrt{\frac{1-k^2u^2}{1-u^2}}\,du=\tfrac{1}{2}\int_0^1 \sqrt{\frac{1-k^2v}{v(1-v)}}\,dv \quad . \quad . \quad . \quad (2)$$

sind analytische Funktionen von $k$ im Innern der von $k=+1$ nach $+\infty$ und von $k=-1$ nach $-\infty$ aufgeschlitzten komplexen $k$-Ebene, die eindeutig bestimmt sind durch die Verabredung

$$|\arg \sqrt{1-k^2v}|<\pi \text{ für } 0\leq v\leq 1$$

(vgl. Proc. Ned. Akad. v. Wetensch., Amsterdam, **45**, 32 (1942) V.E. I, (IV)). Für $k\to 0$ wird dann

$$K(k)\to\frac{\pi}{2}\,;\; E(k)\to\frac{\pi}{2}\,;$$

und für $|k|\leq 1$ gelten die hypergeometrischen Entwicklungen

$$K(k)=\frac{\pi}{2}F(\tfrac{1}{2},\tfrac{1}{2};1;k^2),$$

$$E(k)=\frac{\pi}{2}F(-\tfrac{1}{2},\tfrac{1}{2};1;k^2).$$

(Die Entwicklung für $K(k)$ gilt nicht für $k^2=1$). Zur Berechnung für grosse Werte von $k$ können neben den Methoden der V.E. I (vgl. V.E. I (IV)) die folgenden beiden Sätze benutzt werden, wo die hypergeometrischen Entwicklungen nach $k^2$ in solche nach $\frac{1}{k^2}$, bzw. $1-\frac{1}{k^2}$ transformiert werden, nämlich

**Satz I:** *Für* $|\arg. k^2|<\pi$, $|\arg.\sqrt{1-k^2v}|<\pi$, *und* $0\leq v\leq 1$ *ist*

$$k\,K(k)=K\left(\frac{1}{k}\right)+i\,\operatorname{sgn}.\,I(k^2)\,.\,K\left(\sqrt{1-\frac{1}{k^2}}\right).$$

**Satz II:** *Für* $|\arg. k^2|<\pi$, $|\arg.\sqrt{1-k^2v}|<\pi$, *und* $0\leq v\leq 1$ *ist*

$$k\,E(k)=k^2E\left(\frac{1}{k}\right)+(1-k^2)K\left(\frac{1}{k}\right)+i\,\operatorname{sgn} I(k^2)\,.\left\{K\left(\sqrt{1-\frac{1}{k^2}}\right)-k^2E\left(\sqrt{1-\frac{1}{k^2}}\right)\right\}.$$

Die Funktionen $K\left(\sqrt{1-\frac{1}{k^2}}\right)$ und $E\left(\sqrt{1-\frac{1}{k^2}}\right)$ können für grosses $|k|$ nach den Vorschriften von V.E. I (II), (35) [1]), bzw. V.E. I (III), (47) [2]) mit $l_q=\frac{1}{k}$ in stark konvergente Reihen entwickelt werden.

Obgleich Satz I schon von FUCHS und GOURSAT [3]) abgeleitet worden ist, dürfte der folgende direkte und einfache Beweis vielleicht nicht ohne Interesse sein.

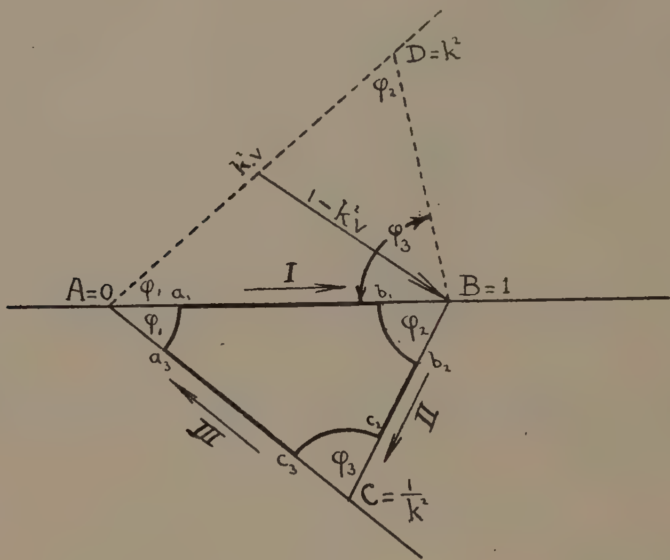


Fig. 1.

*Beweis des Satzes I:*

*a*) $I(k^2)>0$ (Fig. 1).

Wir integrieren

$$\int \frac{dv}{\sqrt{v(1-v)(1-k^2v)}}$$

in negativer Richtung über die Kontur $a_1\, b_1\, b_2\, c_2\, c_3\, a_3\, a_1$.

$$Aa_1=Aa_3=Bb_1=Bb_2=Cc_2=Cc_3=\delta;\; A=0,\, B=1,\, C=\frac{1}{k^2}.$$

Für I: $arg\, v=0$; $arg\;(1-v)=0$ und $arg\;(1-k^2v)$ variiert von 0 bis $-\varphi_3$.

„ II: $arg\, v$ variiert von 0 bis $-\varphi_1$; $arg\;(1-v)=+\varphi_2$ und $arg\;(1-k^2v)=-\varphi_3$.

„ III: $arg\, v=-\varphi_1$; $arg\;(1-v)$ variiert von $+\varphi_2$ nach 0 und $arg\;(1-k^2v)=0$.

Die Integrale über die Kreisbogen $b_1\, b_2$, $c_2\, c_3$ und $a_3\, a_1$ sind $O(\delta^{\frac{1}{2}})$.

Sie verschwinden somit für $\delta\to 0$.

Der Integrand ist analytisch auf der Kontur $a_1 \ldots\ldots a_1$ und innerhalb derselben.

Für I ist das Integral

$$\int_0^1 \frac{dv}{\sqrt{v(1-v)(1-k^2v)}}=2\,K(k) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

[1]) Proc. Ned. Akad. v. Wetensch., Amsterdam, **44**, 1082 (1941).

[2]) Proc. Ned. Akad. v. Wetensch., Amsterdam, **44**, 1203 (1941).

[3]) Cours de M. Hermite 1881—1882. Paris 1883. S. 191—196.

Für II

$$\int_1^{\frac{1}{k^2}} \frac{dv}{\sqrt{v(1-v)(1-k^2v)}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

Setzt man hier

$$v = 1 - z\left(1 - \frac{1}{k^2}\right), \qquad (0 \leqq z \leqq 1),$$

also

$$1 - v = z\left(1 - \frac{1}{k^2}\right),$$

und

$$1 - k^2 v = (1 - k^2)(1 - z),$$

so findet man für II

$$arg\,.\,z = arg\,.\,(1-z) = 0\,;\; arg\,(1-k^2) = -\varphi_3\,;\; arg\left(1 - \frac{1}{k^2}\right) = \varphi_2$$

und (4) geht über in

$$\left.\begin{aligned} & -\int_0^1 \frac{\left(1-\frac{1}{k^2}\right)dz}{\sqrt{z\left(1-\frac{1}{k^2}\right)(1-z)\,(1-k^2)\left\{1-\left(1-\frac{1}{k^2}\right)z\right\}}} \\ & = \frac{-\left|1-\frac{1}{k^2}\right| e^{i\varphi_2}}{\left|1-\frac{1}{k^2}\right|^{\frac{1}{2}} |1-k^2|^{\frac{1}{2}}\, e^{\frac{i(\varphi_2-\varphi_3)}{2}}} \int_0^1 \frac{dz}{\sqrt{z(1-z)\left\{1-\left(1-\frac{1}{k^2}\right)z\right\}}} \\ & = -\frac{2e^{\frac{i(\varphi_2+\varphi_3)}{2}}}{|k|} K\left(\sqrt{1-\frac{1}{k^2}}\right) = -\frac{2i}{k} K\left(\sqrt{1-\frac{1}{k^2}}\right). \end{aligned}\right\} \quad (5)$$

Für III ist das Integral

$$\int_{\frac{1}{k^2}}^{0} \frac{dv}{\sqrt{v(1-v)(1-k^2v)}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

Setzt man hier $v = \frac{w}{k^2}$, $(0 \leqq w \leqq 1)$, so geht (6) über in

$$-\frac{1}{k^2}\int_0^1 \frac{dw}{\sqrt{\frac{w}{k^2}\left(1-\frac{w}{k^2}\right)(1-w)}} = -\frac{2}{k} K\left(\frac{1}{k}\right). \quad . \; . \; . \quad (7)$$

Aus (3), (5) und (7) ergibt sich

$$\underline{k\,K(k) = K\left(\frac{1}{k}\right) + i\,K\left(\sqrt{1-\frac{1}{k^2}}\right) \text{ für } I(k^2) > 0.} \quad . \quad . \quad (8)$$

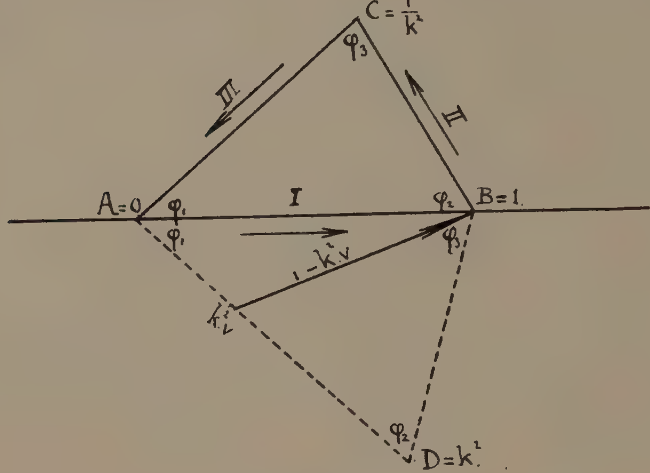


Fig. 2.

*b*) $I(k^2) < 0$ (Fig. 2).

In analoger Weise findet man bei Integration in positiver Richtung über die Kontur $ABC$:

für I: $arg\, v = 0$; $arg\,(1-v) = 0$ und $arg\,(1-k^2 v)$ variiert von 0 bis $+\varphi_3$;

„ II: $arg\, v$ variiert von 0 bis $+\varphi_1$; $arg\,(1-v) = -\varphi_2$ und $arg\,(1-k^2 v) = +\varphi_3$;

„ III: $arg\, v = +\varphi_1$; $arg\,(1-v)$ variiert von $-\varphi_2$ nach 0 und $arg\,(1-k^2 v) = 0$.

Wie oben ist das Integral für I

$$\int\limits_0^1 \frac{dv}{\sqrt{v(1-v)(1-k^2 v)}} = 2K(k). \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Für II ist $arg\,(1-k^2) = +\varphi_3$; $arg\left(1-\frac{1}{k^2}\right) = -\varphi_2$ und das Integral hat den Wert

$$\frac{-\left|1-\frac{1}{k^2}\right| e^{-i\varphi_2}}{\left|1-\frac{1}{k^2}\right|^{\frac{1}{2}} \left|1-k^2\right|^{\frac{1}{2}} e^{\frac{-i(\varphi_2-\varphi_3)}{2}}} \int\limits_0^1 \frac{dz}{\sqrt{z(1-z)\left\{1-\left(1-\frac{1}{k^2}\right)z\right\}}} = \frac{+2i}{k} K\left(\sqrt{1-\frac{1}{k^2}}\right). \quad (10)$$

Schliesslich für III bekommen wir

$$\int\limits_{\frac{1}{k^2}}^0 \frac{dv}{\sqrt{v(1-v)(1-k^2 v)}} = -\frac{2}{k} K\left(\frac{1}{k}\right). \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Aus (9), (10) und (11) ergibt sich

$$\underline{k\,K(k) = K\left(\frac{1}{k}\right) - i\,K\left(\sqrt{1-\frac{1}{k^2}}\right) \text{ für } I(k^2) < 0.} \quad . \quad . \quad (12)$$

Aus (8) und (12) folgt Satz I.

*Beweis des Satzes II.* Bekanntlich ist

$$E(k) = (1-k^2)\,K(k) + k\,(1-k^2)\,\frac{d\,K(k)}{d\,k} = (1-k^2)\,\frac{d\,k\,.\,K(k)}{d\,k}. \quad . \quad . \quad (13)$$

Setzt man

$$k^2 = c$$

$$K(k) = \bar{K}(c); \qquad E(k) = \bar{E}(c);$$

$$K\left(\frac{1}{k}\right) = K\left(\frac{1}{c}\right); \qquad E\left(\frac{1}{k}\right) = E\left(\frac{1}{c}\right);$$

$$K\left(\sqrt{1-\frac{1}{k^2}}\right) = K\left(\frac{c-1}{c}\right); \quad E\left(\sqrt{1-\frac{1}{k^2}}\right) = E\left(\frac{c-1}{c}\right),$$

so geht Satz I über in

$$k\,.\,\bar{K}(c) = \bar{K}\left(\frac{1}{c}\right) + i\,sgn\,.\,I(c)\,.\,\bar{K}\left(\frac{c-1}{c}\right), \quad . \quad . \quad . \quad (14)$$

und (13) in

$$\bar{E}(c) = 2\,k\,(1-c)\,\frac{d\,.\,k\,.\,\bar{K}(c)}{d\,c} = 2\,c\,(1-c)\,\frac{d\,\bar{K}(c)}{d\,c} + (1-c)\,\bar{K}(c) \quad . \quad (15)$$

Aus (14) und (15) ergibt sich

$$\bar{E}(c) = 2\,k\,(1-c)\,\frac{d}{d\,c}\left\{\bar{K}\left(\frac{1}{c}\right) + i\,sgn\,.\,I(c)\,.\,\bar{K}\left(\frac{c-1}{c}\right)\right\} =$$

$$= 2\,k\,(1-c)\left\{-\frac{1}{c^2}\left(\frac{d\,\bar{K}(u)}{d\,u}\right)_{u=\frac{1}{c}} + \frac{i\,sgn\,I(c)}{c^2}\left(\frac{d\,\bar{K}(u)}{d\,u}\right)_{u=\frac{c-1}{c}}\right\} =$$

$$= -2k(1-c)\left\{\frac{\bar{E}\left(\frac{1}{c}\right) - \left(1-\frac{1}{c}\right)\bar{K}\left(\frac{1}{c}\right)}{2\,(c-1)} - i\,sgn\,.\,I(c)\,.\,\frac{\bar{E}\left(\frac{c-1}{c}\right) - \frac{1}{c}\,.\,\bar{K}\left(\frac{c-1}{c}\right)}{2\,(c-1)}\right\} =$$

$$= k\left\{\bar{E}\left(\frac{1}{c}\right) - \left(1-\frac{1}{c}\right)\bar{K}\left(\frac{1}{c}\right)\right\} - i\,k\,sgn\,.\,I(c)\left\{\bar{E}\left(\frac{c-1}{c}\right) - \frac{1}{c}\,\bar{K}\left(\frac{1-c}{c}\right)\right\},$$

also

$$k\,E(k) = k^2\,E\left(\frac{1}{k}\right) + (1-k^2)\,K\left(\frac{1}{k}\right) + i\,sgn\,.\,I(k^2)\left\{K\left(\sqrt{1-\frac{1}{k^2}}\right) - k^2\,E\left(\sqrt{1-\frac{1}{k^2}}\right)\right\}$$

w. z. b. w.

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**Mathematics.** — *Contribution à la théorie métrique des approximations diophantiques non-linéaires* (Première communication). Par J. F. KOKSMA. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of January 31, 1942.)

§ 1. *Introduction.*

I. On doit à M. A. KHINTCHINE le théorème important suivant [1]:

**Théorème 1.** *Soit $n$ un nombre naturel et $\omega(x)$ une fonction positive du nombre naturel $x$, telle que la fonction $x(\omega(x))^n$ tend vers zéro monotonement, si $x \to \infty$. Considérons les inégalités simultanées*

$$|\theta_\nu x - y_\nu| < \omega(x) \qquad (\nu = 1, 2, \ldots, n), \quad \ldots \quad (1)$$

*où $(\theta_1, \theta_2, \ldots, \theta_n)$ désigne un point quelconque de l'espace $R_n$. Considérons en outre la série*

$$\sum_{x=1}^{\infty} (\omega(x))^n . \quad \ldots \quad (2)$$

**Assertion A.** *Pour presque tous les points $(\theta_1, \theta_2, \ldots, \theta_n)$ de l'espace $R_n$ le système (1) admet une infinité de solutions entières $x \geqq 1, y_1, y_2, \ldots, y_n$, si la série (2) diverge.*

**Assertion B.** *Pour presque tous les points $(\theta_1, \theta_2, \ldots, \theta_n)$ de l'espace $R_n$ le système (1) n'a qu'un nombre fini de solutions entières $x \geqq 1, y_1, y_2, \ldots, y_n$, si la série (2) converge.*

**Remarques.** 1. L'expression „presque tous les points" etc. veut dire que les autres points forment un ensemble de mesure nulle au sens de LEBESGUE.

2. M. KHINTCHINE considère aussi l'intégral $\int^{\infty} (\omega(t))^n \, dt$ au lieu de la somme (2) sous la condition supplémentaire que la fonction $\omega(t)$ soit continue. Il est clair que cela ne fait aucune différence essentielle.

II. Soit $f_\nu(1), f_\nu(2), \ldots$ pour $\nu = 1, 2, \ldots, n$ une suite croissante arbitraire de nombres naturels. Alors le théorème 1A ne donne aucun résultat sur l'approximation simultanée des $n$ expressions

$$\theta_\nu f_\nu(x) - y_\nu \qquad (\nu = 1, 2, \ldots, n)$$

à zéro, excepté si $f_\nu(x) = x$ $(x = 1, 2, \ldots)$, tandis que le résultat fourni par le théorème 1B est peu intéressant.

Toutefois dans le cas général indiqué on peut présumer un théorème analogue. Comme je l'ai démontré à une autre occasion, la généralisation de l'assertion $B$ ne rencontre aucune difficulté. Par exemple on a la proposition suivante, contenue dans une proposition encore plus générale [2]:

**Théorème 2.** *Soit $n$ un nombre naturel, $f_\nu(1), f_\nu(2), \ldots$ pour $\nu = 1, 2, \ldots, n$ une suite croissante de nombres naturels et $\omega_\nu(x)$ pour $\nu = 1, 2, \ldots, n$ une fonction positive du nombre naturel $x$, telle que les séries $\sum_{x=1}^{\infty} \operatorname{Max}_{1 \leq \nu \leq n} (\omega_\nu(x)/f_\nu(x))$ et*

$$\sum_{x=1}^{\infty} \prod_{\nu=1}^{n} \omega_\nu(x) \quad \ldots \quad (3)$$

[1] A. KHINTCHINE, Zur metrischen Theorie der diophantischen Approximationen. Math. Z. **24**, p. 706—714 (1926).

[2] J. F. KOKSMA, Metrisches über die Approximation reeller Zahlen. Proc. Ned. Akad. v. Wetensch., Amsterdam, **41**, p. 45—47 (1938).

Ueber die asymptotische Verteilung eines beliebigen Systems $(f_\nu)$ von $n$ reellen Funktionen $f_\nu$ der $m$ ganzzahligen Veränderlichen $x_1, x_2, \ldots, x_m$ modulo Eins. Proc. Ned. Akad. v. Wetensch. **43**, p. 211—214 (1940).

*convergent. Alors pour presque tous les points* $(\vartheta_1, \vartheta_2, \ldots, \vartheta_n)$ *de l'espace* $R_n$ *le système des inégalités*

$$|\theta_\nu f_\nu(x) - y_\nu| < \omega_\nu(x) \qquad (\nu = 1, 2, \ldots, n) \quad . \quad . \quad (4)$$

*n'a qu'un nombre fini de solutions entières* $x \geqq 1, y_1, y_2, \ldots, y_n$.

Jusqu'ici je n'ai pas réussi à généraliser d'une manière aussi générale la belle démonstration de M. KHINTCHINE du théorème 1A. Mais pourtant les résultats obtenus concernent quelques classes bien étendues de systèmes de suites $f_\nu(1), f_\nu(2), \ldots,$ comme montrent déjà les théorèmes 3 et 4, dont la démonstration forme le but de cette première communication. Autrefois j'ai donné un exposé succinct des résultats de ces recherches [3]).

III. Soit

$$f(1), f(2), \ldots$$

une suite arbitraire de nombres naturels croissants. Par $d(z, x)$ j'indique le plus grand diviseur commun $(f(z), f(x))$ des nombres $f(z)$ et $f(x)$ $(z, x = 1, 2, \ldots)$. Alors pour toute suite de cette espèce et toute paire de nombres $\alpha, \beta$ satisfaisant aux inégalités $0 \leqq \alpha < \beta \leqq 1$, je définis la fonction arithmétique $A(x, \alpha, \beta)$ $(x = 1, 2, \ldots)$ comme le nombre des nombres naturels de l'intervalle $\alpha f(x) < u < \beta f(x)$ qui pour aucune valeur de l'indice $z$ $(1 \leqq z < x)$ ne sont divisibles par le nombre $\dfrac{f(x)}{d(z, x)}$.

**Définition 1.** *Nous dirons que la suite* $f(1), f(2), \ldots$ *de nombres naturels croissants possède la propriété* $\mathfrak{P}$, *si à toute paire de nombres* $\alpha, \beta$ $(0 \leqq \alpha < \beta \leqq 1)$ *correspondent un nombre positif* $C = C(\alpha, \beta)$ *et un indice* $x_0$ *indépendant de* $x$ *tels que*

$$A(x, \alpha, \beta) \geqq C(\beta - \alpha) f(x), \text{ si } x \geqq x_0.$$

*Nous dirons que cette suite possède la propriété* $\mathfrak{P}^*$, *si en outre il est possible de choisir le nombre* $C$ *indépendant des nombres* $\alpha, \beta$.

Soit $S$ un système de $n \geqq 1$ suites croissantes de nombres naturels

$$f_\nu(1), f_\nu(2), \ldots \qquad (\nu = 1, 2, \ldots, n),$$

$B$ un système de $n$ paires de nombres $\alpha_\nu, \beta_\nu$ avec $0 \leqq \alpha_\nu < \beta_\nu \leqq 1$ et soit posé pour tout nombre naturel $N$

$$H(N, B) = H(N, B, S) = \frac{1}{N} \sum_{x=1}^{N} \prod_{\nu=1}^{n} \frac{A_\nu(x, \alpha_\nu, \beta_\nu)}{f_\nu(x)}, \quad . \quad . \quad . \quad (5)$$

où $A_\nu(x, \alpha_\nu, \beta_\nu)$ indique la fonction $A(x, \alpha_\nu, \beta_\nu)$ correspondante à la suite $f_\nu(1), f_\nu(2), \ldots$.

**Définition 2.** *Nous dirons que le système* $S$ *possède la propriété* $Q$, *si à tout système* $B$ *correspondent un nombre positif* $c = c(B)$ *et un indice* $N_0$ *indépendant de* $N$ *tels que*

$$\frac{H(N, B, S)}{\prod_{\nu=1}^{n} (\beta_\nu - \alpha_\nu)} \geqq c(B), \text{ si } N \geqq N_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

*Nous dirons que* $S$ *possède la propriété* $Q^*$, *si en outre il est possible de choisir le nombre* $c$ *indépendant du système* $B$.

*Si un système* $S$ *formé par une suite seule possède la propriété* $Q$ *(la propriété* $Q^*$*) nous dirons que cette suite elle-même possède la propriété* $Q$ *(la propriété* $Q^*$*).*

**Conclusions évidentes.** 1. Si une suite possède la propriété $\mathfrak{P}$ (la propriété $\mathfrak{P}^*$) elle possède la propriété $Q$ (la propriété $Q^*$).

2. Si toute suite d'un système $S$ possède la propriété $\mathfrak{P}$ (la propriété $\mathfrak{P}^*$) le système $S$ possède la propriété $Q$ (la propriété $Q^*$).

3. Si à un système $S$ possédant la propriété $Q$ (la propriété $Q^*$) on ajoute une suite possédant la propriété $\mathfrak{P}$ (la propriété $\mathfrak{P}^*$), le nouveau système $S'$ possède la propriété $Q$ (la propriété $Q^*$).

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[3]) J. F. KOKSMA, Niet-lineaire simultane approximaties. Handelingen 28ste Ned. Nat. Gen. Congres, p. 95—96 (1941).

4. Si les $n$ suites d'un système $S$ sont identiques deux à deux, et possèdent la propriété $Q$ (la propriété $Q^*$), la généralisation de HÖLDER de l'inégalité de CAUCHY-RIEMANN-SCHWARZ [4]) nous apprend que le système $S$ possède la propriété $Q$ (la propriété $Q^*$) sous la condition supplémentaire qu'on se restreigne aux systèmes $B$ avec $\alpha_\nu = \alpha_1$, $\beta_\nu = \beta_1$.

**Exemples.** A. Il est clair que la suite $f(1)$, $f(2)$, ... possède la propriété $\mathfrak{P}^*$, si

a. $f(x) = d^x$, où $d$ désigne un nombre naturel $\geqq 2$, indépendant de $x$ (on peut prendre pour $C$ toute constante positive $< \frac{d-1}{d}$),

ou b. $f(x) = x\,!$, (dans les cas $b$, $c$ et $d$ on peut prendre pour $C$ toute constante positive $< 1$),

ou c. $f(x) = p(x) =$ le $x$-ième nombre premier,

ou d. $f(x) = p(1)\,p(2)\dots p(x)$.

B. Plus généralement, je vais démontrer que toute suite qui possède la propriété

$$\operatorname*{Lim\,sup}_{x\to\infty} \sum_{z=1}^{x-1} \frac{f(z)}{f(x)} < 1 \quad \dots\dots\dots \quad (7)$$

possède la propriété $\mathfrak{P}^*$. En effet, on a

$$A(x, \alpha, \beta) \geqq [(\beta-\alpha)\,f(x)] - \sum_{z=1}^{x-1} [(\beta-\alpha)\,d(z, x)] - (x-1)$$

$$\geqq (\beta-\alpha)\,f(x) \left\{ \left(1 - \sum_{z=1}^{x-1} \frac{f(z)}{f(x)}\right) - \frac{x}{(\beta-\alpha)\,f(x)} \right\},$$

d'où suit l'assertion parceque (7) entraîne $\frac{x}{f(x)} \to 0$ si $x \to \infty$.

**Application.** Toute suite qui possède la propriété

$$f(x) \geqq \mu f(x-1) \text{ pour } x \geqq x_0,$$

$\mu$ désignant un nombre $> 2$ indépendant de $x$, et $x_0$ désignant un indice convenablement choisi, possède la propriété $\mathfrak{P}^*$. En effet, on voit immédiatement que l'inégalité (7) est remplie.

C. On peut démontrer que la suite $1, 2, 3, \dots$ de tous les nombres naturels (donc $f(x) = x$) possède la propriété $Q^*$ et aussi que pour tout système d'exposants naturels $m_1, m_2, \dots, m_n$ le système des $n$ suites définies par $f_\nu(x) = x^{m_\nu}$ $(\nu = 1, 2, \dots, n)$ possède la propriété $Q^*$. Comme dans la première communication je ne ferai pas usage de cette assertion, je n'en donne pas une démonstration ici.

Remarquons finalement que d'après la conclusion 3 chaque système $S$, formé par $n$ suites nommées dans les exemples A, B, C possédera la propriété $Q^*$. D'ailleurs chacune de ces suites y peut configurer autant de fois qu'on le veut.

IV. Formulons maintenant les Théorèmes 3 et 4.

**Théorème 3.** *Soit $n$ un nombre naturel et $\omega_\nu(x)$ pour $\nu = 1, 2, \dots, n$ une fonction positive non-croissante du nombre naturel $x$, telle que la série (3) diverge et satisfaisant à*

$$\omega_\nu(x) \leqq \tfrac{1}{2}\ (\nu = 1, 2, \dots, n)\ (x \geqq 1);\ x \prod_{\nu=1}^{n} \omega_\nu(x) \to 0, \text{ si } x \to \infty. \quad (8)$$

*Soit $S$ un système de $n$ suites croissantes de nombres naturels $f_\nu(1), f_\nu(2), \dots$ possédant la propriété $Q$. Alors l'ensemble des points $(\theta_1, \theta_2, \dots, \theta_n)$ de l'espace $R_n$ pour lesquels le système des inégalités (4) admet une infinité de solutions entières $x \geqq 1, y_1, y_2, \dots, y_n$*

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[4]) Il suffit de prendre le cas special

$$\left\{ \sum_{x=1}^{N} c_x \right\}^n \leqq N^{n-1} \sum_{x=1}^{N} c_x^n. \qquad (N \geqq 1,\ n \geqq 1.\ c_1, \dots, c_N \geqq 0).$$

*est épais partout, c'est à dire: possède par rapport à chaque parallélépipède* $\alpha_\nu < u_\nu < \beta_\nu$ $(\nu = 1, 2, \ldots, n)$ *une densité positive au sens de* LEBESGUE[5]).

**Théorème 4.** *Soit* $n$ *un nombre naturel et* $\omega_\nu(x)$ *pour* $\nu = 1, 2, \ldots, n$ *une fonction positive non-croissante du nombre naturel* $x$, *telle que la série* (3) *diverge et satisfaisant à* (8). *Soit finalement* $S$ *un système de* $n$ *suites croissantes de nombres naturels* $f_\nu(1), f_\nu(2), \ldots$, *possédant la propriété* $Q^*$ *et tel que*

$$d_\nu(z, x) \to \infty, \text{ si } z \to \infty, \text{ uniformément en } x > z, \ (\nu = 1, 2, \ldots, n).$$

*Alors pour presque tous les points* $(\theta_1, \theta_2, \ldots, \theta_n)$ *de l'espace* $R_n$ *le système des inégalités* (4) *a une infinité de solutions entières* $x \geqq 1, y_1, y_2, \ldots, y_n$.

**Remarque.** Il saute aux yeux qu'en général les relations (8) ne donnent aucune difficulté dans les applications, car les assertions des théorèmes 3 et 4 ayant été démontrés dans le cas (8), restent justes, si l'on y remplace les fonctions $\omega_\nu(x)$ par des fonctions ayant des valeurs supérieures.

§ 2. *Lemmes.*

**Remarques préliminaires.** 1. Si $S$ désigne un système de $n$ suites croissantes de nombres naturels $f_\nu(1), f_\nu(2), \ldots$, nous indiquerons par $f^*_{\nu k}(x+i)$ et $f^*_{\nu i}(x+k)$ les quotients

$$\frac{f_\nu(x+i)}{d_\nu(x+i, x+k)} \quad \text{et} \quad \frac{f_\nu(x+k)}{d_\nu(x+i, x+k)} \ (0 \leqq i < k),$$

où $d_\nu(x+i, x+k)$ désigne le plus grand diviseur commun des nombres $f_\nu(x+i)$ et $f_\nu(x+k)$, comme il a été convenu dans § 1. III (les conventions du § 1. III restent valables pendant toute la durée de la démonstration et donc aussi dans les lemmes 1, 2, 3).

2. Sans nuire à la généralité on peut se restreindre aux points $(\theta_1, \theta_2, \ldots, \theta_n)$ du cube $0 < u_\nu < 1$ $(\nu = 1, 2, \ldots, n)$.

**Lemme 1.** *Soit* $S$ *un système de* $n$ *suites croissantes de nombres naturels* $f_\nu(1), f_\nu(2), \ldots, B$ *un système de* $2n$ *nombres* $\alpha_\nu, \beta_\nu$ *tels que* $0 \leqq \alpha_\nu < \beta_\nu \leqq 1$, $\omega_\nu(x)$ *une fonction non-croissante et positive de* $x = 1, 2, \ldots$ *et* $P^{(x)}_{r_1, r_2, \ldots, r_n}$

$$(0 \leqq r_\nu < f_\nu(x) \quad (\nu = 1, 2, \ldots, n);\ x = 1, 2, \ldots)$$

*le parallélépipède à* $n$ *dimensions*

$$\left| u_\nu - \frac{r_\nu}{f_\nu(x)} \right| < \frac{\omega_\nu(x)}{f_\nu(x)} \quad (\nu = 1, 2, \ldots, n).$$

*Alors parmi les parallélépipèdes* $P^{(x+k)}_{r_1, r_2, \ldots, r_n}$ $(x, k \geqq 1$ *étant supposés fixes) il y a*

$$\left[ 4^n \prod_{\nu=1}^{n} (\beta_\nu - \alpha_\nu) \prod_{\nu=1}^{n} f_\nu(x+k) \sum_{i=0}^{k-1} \prod_{\nu=1}^{n} \left( 1 + \frac{1}{(\beta_\nu - \alpha_\nu)\, d_\nu(x+i, x+k)} \right) \prod_{\nu=1}^{n} \omega_\nu(x+i) \right] \quad (9)$$

*au plus qui pour au moins un des parallélépipèdes* $P^{(x+i)}_{s_1, s_2, \ldots, s_n}$ $(0 \leqq i < k)$ *possèdent les propriétés suivantes:*

$P^{(x+k)}_{r_1, r_2, \ldots, r_n}$ *et* $P^{(x+i)}_{s_1, s_2, \ldots, s_n}$ *ont des points communs;*

$$s_\nu f^*_{\nu i}(x+k) - r_\nu f^*_{\nu k}(x+i) \neq 0 \qquad (\nu = 1, 2, \ldots, n); \quad \ldots \quad (10)$$

et

$$\alpha_\nu f_\nu(x+k) < r_\nu < \beta_\nu f_\nu(x+k) \qquad (\nu = 1, 2, \ldots, n). \quad \ldots \quad (11)$$

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5) Dans ce mémoire $(u_1, u_2, \ldots, u_n)$ désignera le point variable de l'espace $R_n$.

**Démonstration.** Si les parallélépipèdes $P^{(x+i)}_{r_1, r_2, \ldots, r_n}$ et $P^{(x+k)}_{s_1, s_2, \ldots, s_n}$ ont des points communs, on a nécessairement

$$\left| \frac{s_\nu}{f_\nu(x+i)} - \frac{r_\nu}{f_\nu(x+k)} \right| < \frac{\omega_\nu(x+i)}{f_\nu(x+i)} + \frac{\omega_\nu(x+k)}{f_\nu(x+k)} \qquad (\nu = 1, 2, \ldots, n),$$

c'est à dire

$$| s_\nu f_\nu(x+k) - r_\nu f_\nu(x+i) | < 2\,\omega_\nu(x+i) f_\nu(x+k) \qquad (\nu = 1, 2, \ldots, n),$$

et donc :

$$| s_\nu f^*_{\nu i}(x+k) - r_\nu f^*_{\nu k}(x+i) | < 2\,\omega_\nu(x+i) f^*_{\nu i}(x+k) \quad (\nu = 1, 2, \ldots, n). \quad (12)$$

Or, si en outre les $n$ inégalités (10) sont valables, il y a un système de $n$ nombres entiers $K_\nu$, tel que

$$1 \leqq | K_\nu | \leqq 2\,\omega_\nu(x+i) f^*_{\nu i}(x+k) \qquad (\nu = 1, 2, \ldots, n), \; . \; . \quad (13)$$

$$s_\nu f^*_{\nu i}(x+k) - r_\nu f^*_{\nu k}(x+i) = K_\nu \qquad (\nu = 1, 2, \ldots, n). \; . \; . \quad (14)$$

Les nombres $x$, $k$, $i$, $\nu$, $K_\nu$ étant supposés fixes, toutes les solutions entières $X$, $Y$ de l'équation diophantique

$$X f^*_{\nu i}(x+k) - Y f^*_{\nu k}(x+i) = K_\nu \quad . \; . \; . \; . \quad (15)$$

s'expriment par les formules

$$X = K_\nu X_0 + h f^*_{\nu k}(x+i), \quad Y = K_\nu Y_0 + h f^*_{\nu i}(x+k) \quad (h = 0, \pm 1, \pm 2, \ldots),$$

où $X_0$, $Y_0$ désigne une solution entière quelconque de l'équation

$$X f^*_{\nu i}(x+k) - Y f^*_{\nu k}(x+i) = 1.$$

Ainsi, à cause de la relation

$$d_\nu(x+i, x+k) = \frac{f_\nu(x+k)}{f^*_{\nu i}(x+k)},$$

le nombre des solutions entières $X$, $Y$ de (15) satisfaisant à la fois à la relation

$$\alpha_\nu f_\nu(x+k) < Y < \beta_\nu f_\nu(x+k) \quad . \; . \; . \; . \; . \; . \quad (16)$$

sera au plus

$$[(\beta_\nu - \alpha_\nu) d_\nu(x+i, x+k)] + 1.$$

Alors, si $K_\nu$ parcourt les valeurs entières données par (13), le nombre total des solutions entières $X$, $Y$ de (15) et (16) sera au plus

$$4\,\omega_\nu(x+i) f^*_{\nu i}(x+k) \{(\beta_\nu - \alpha_\nu) d_\nu(x+i, x+k) + 1\} =$$

$$= 4 \left( 1 + \frac{1}{(\beta_\nu - \alpha_\nu) d_\nu(x+i, x+k)} \right) (\beta_\nu - \alpha_\nu)\, \omega_\nu(x+i) f_\nu(x+k).$$

De cela il découle que le nombre total des $P^{(x+k)}_{r_1, r_2, \ldots, r_n}$ satisfaisant pour au moins un des $P^{(x+i)}_{s_1, s_2, \ldots, s_n}$ $(0 \leqq i < k)$ aux inégalités (12), (10) et (11) est au plus égal à (9). C.q.f.d.

**Lemme 2.** *Soient les systèmes $S$ et $B$, les fonctions $\omega_\nu(x)$ et les parallélépipèdes $P^{(x)}_{r_1, r_2, \ldots, r_n}$ définis comme dans le lemme 1 et soit en outre $\omega_\nu(x) \leqq \frac{1}{2}$. $E_x(B)$ désigne la somme de tous ces $P^{(x)}_{r_1, r_2, \ldots, r_n}$ ($x$ étant supposé fixe) dont le centre est situé dans le parallélépipède*

$$\alpha_\nu < u_\nu < \beta_\nu \qquad (\nu = 1, 2, \ldots, n).$$

*Enfin $F_{x,k}(B)$ $(k \geqq 0)$ désigne l'ensemble de ceux des points de $E_{x+k}(B)$ qui n'appartiennent à aucun des ensembles $E_x(B)$, $E_{x+1}(B), \ldots, E_{x+k-1}(B)$ $(F_{x,0}(B) = E_x(B))$.*

*Alors pour toute paire de nombres entiers* $x \geqq 1, k \geqq 0$ *la mesure au sens de* LEBESGUE *de l'ensemble* $F_{x,k}(B)$ *satisfait à l'inégalité*

$$m F_{x,k}(B) \geqq 2^n \prod_{\nu=1}^{n} \omega_\nu(x+k) \left\{ \frac{\prod_{\nu=1}^{n} A_\nu(x+k, \alpha_\nu, \beta_\nu)}{\prod_{\nu=1}^{n} f_\nu(x+k)} \right.$$

$$\left. - 4^n \prod_{\nu=1}^{n} (\beta_\nu - \alpha_\nu) \sum_{i=0}^{k-1} \prod_{\nu=1}^{n} \left(1 + \frac{1}{(\beta_\nu - \alpha_\nu) d_\nu(x+i, x+k)}\right) \prod_{\nu=1}^{n} \omega_\nu(x+i) \right\}.$$

**Démonstration.** Toutes les solutions entières $X, Y$ de l'équation

$$X f^*_{\nu,i}(x+k) - Y f^*_{\nu,k}(x+i) = 0$$

($x, k, i, \nu$ étant supposés fixes) s'expriment par les formules

$$X = h f^*_{\nu,k}(x+i), \quad Y = h f^*_{\nu,i}(x+k) \qquad (h = 0, \pm 1, \pm 2, \ldots),$$

c'est à dire que l'inegalité

$$s_\nu f^*_{\nu,i}(x+k) - r_\nu f^*_{\nu,k}(x+i) \neq 0 \quad . \quad . \quad . \quad (17)$$

est valable pour tout $r_\nu$ qui n'est pas divisible par $f^*_{\nu,i}(x+k)$. Ainsi, $x, k, \nu$ étant supposés fixes, le nombre des $r_\nu$ de l'intervalle $\alpha_\nu f_\nu(x+k) < r_\nu < \beta_\nu f_\nu(x+k)$ qui satisfont pour tout $P^{(x+i)}_{s_1, s_2, \ldots, s_n}$ $(0 \leqq i < k)$ à l'inégalité (17), est au moins égal à $A_\nu(x+k, \alpha_\nu, \beta_\nu)$. De là nous tirons la conclusion que le nombre des $P^{(x+k)}_{r_1, r_2, \ldots, r_n}$ satisfaisant pour tout $P^{(x+i)}_{s_1, s_2, \ldots, s_n}$ $(0 \leqq i < k)$ aux $n$ inégalités (10) et dont en outre le centre est situé dans le parallélépipède $\alpha_\nu < u_\nu < \beta_\nu$ $(\nu = 1, 2, \ldots, n)$ est au moins égal à

$$\prod_{\nu=1}^{n} A_\nu(x+k, \alpha_\nu, \beta_\nu).$$

D'après le lemme 1, le nombre des $P^{(x+k)}_{r_1, r_2, \ldots, r_n}$, satisfaisant pour tout $P^{(x+i)}_{s_1, s_2, \ldots, s_n}$ aux $n$ inégalités (10), n'ayant de points communs avec aucun des $P^{(x+i)}_{s_1, s_2, \ldots, s_n}$ et dont le centre est situé dans le parallélépipède $\alpha_\nu < u_\nu < \beta_\nu$ $(\nu = 1, 2, \ldots, n)$ est donc au moins égal à

$$\prod_{\nu=1}^{n} A_\nu(x+k, \alpha_\nu, \beta_\nu) - 4^n \prod_{\nu=1}^{n} (\beta_\nu - \alpha_\nu) \prod_{\nu=1}^{n} f_\nu(x+k) \sum_{i=0}^{k-1} \prod_{\nu=1}^{n} \left(1 + \frac{1}{(\beta_\nu - \alpha_\nu) d_\nu(x+i, x+k)}\right) \prod_{\nu=1}^{n} \omega_\nu(x+i).$$

Or, ceci prouve le lemme à cause de la définition de l'ensemble $F_{x,k}(B)$, les $P^{(x+k)}_{r_1, r_2, \ldots, r_n}$ n'ayant de points communs deux à deux et possédant tous le volume

$$\frac{2^n \prod_{\nu=1}^{n} \omega_\nu(x+k)}{\prod_{\nu=1}^{n} f_\nu(x+k)}.$$

**Lemme 3.** *Si les conditions du lemme 2 et* (8) *sont remplies et si en outre les inégalités* (6) *et*

$$\prod_{\nu=1}^{n} \left(1 + \frac{1}{(\beta_\nu - \alpha_\nu) d_\nu(x+i, x+k)}\right) \leqq \Omega(B) \quad (x \geqq x_0; i = 0, 1, \ldots, k-1; k \geqq 1) \quad (18)$$

*sont vérifiées, où* $N_0$, $c(B)$, $x_0$, $\Omega(B)$ *désignent des nombres positifs ne dépendant que de la définition des systèmes* $S$ *et* $B$ *(et donc indépendant de* $N, x, k, i$*), alors à tout indice* $x \geqq 1$ *un nombre naturel* $K = K(x)$ *correspond tel que*

$$m \sum_{i=0}^{K-1} E_{x+i}(B) \geqq \frac{(c(B))^2}{12.2^n \, \Omega(B)} \prod_{\nu=1}^{n} (\beta_\nu - \alpha_\nu) . \quad . \quad . \quad . \quad . \quad (19)$$

**Remarque.** Il est clair que sans nuire à la généralité il suffit de démontrer (19) sous

la condition moins exigeante $x \geqq X_0$, où $X_0$ désigne un nombre positif dépendant de $S, B$, et du choix des $\omega_\nu(x)$. Car alors le lemme est évident.

**Démonstration.** Comme $F_{x,k}(B) \subseteq E_{x+k}(B)$, on a

$$m\left(\sum_{i=0}^{K-1} E_{x+i}(B)\right) \geqq m\left(\sum_{k=0}^{K-1} F_{x,k}(B)\right) = \sum_{k=0}^{K-1} m\, F_{x,k}(B),$$

les ensembles $F_{x,k}(B)$ n'ayant pas de points communs deux á deux. Je remarque que'il est possible de choisir le nombre $X_0' \geqq x_0$, tel que

$$\prod_{\nu=1}^{n} \omega_\nu(x) < \frac{c(B)}{2 \cdot 4^n \cdot \Omega(B)} \text{ pour } x \geqq X_0',$$

car d'après (8) le membre gauche tend vers zéro, si $x \to \infty$. De la divergence de la série (3) nous concluons qu'un nombre $K = K(x)$ existe tel que

$$\sum_{k=0}^{K-1} \prod_{\nu=1}^{n} \omega_\nu(x+k) \leqq \frac{c(B)}{2 \cdot 4^n \cdot \Omega(B)} < \sum_{k=0}^{K} \prod_{\nu=1}^{n} \omega_\nu(x+k) \text{ pour } x \geqq X_0'. \quad (20)$$

Alors du lemme 2 et de (18) nous tirons l'inégalité

$$m\, F_{x,k}(B) \geqq 2^n \prod_{\nu=1}^{n} \omega_\nu(x+k) \left\{ \frac{\prod_{\nu=1}^{n} A_\nu(x+k, \alpha_\nu, \beta_\nu)}{\prod_{\nu=1}^{n} f_\nu(x+k)} - \frac{c(B)}{2} \prod_{\nu=1}^{n} (\beta_\nu - \alpha_\nu) \right\} (0 \leqq k < K).$$

D'après (5) on a $\displaystyle\sum_{x=1}^{j} \prod_{\nu=1}^{n} \frac{A_\nu(x, \alpha_\nu, \beta_\nu)}{f_\nu(x)} = j\, H(j, B) \qquad (j \geqq 1)$

et donc en vertu de (20)

$$\left.\begin{aligned}
&\sum_{k=0}^{K-1} m\, F_{x,k}(B) \geqq 2^n \sum_{k=0}^{K-1} \prod_{\nu=1}^{n} \omega_\nu(x+k) \{(x+k) H(x+k, B) - (x+k-1) H(x+k-1, B)\} \\
&- \frac{(c(B))^2}{4 \cdot 2^n \Omega(B)} \prod_{\nu=1}^{n} (\beta_\nu - \alpha_\nu) = 2^n \sum_{k=0}^{K-1} (x+k) H(x+k, B) \left\{ \prod_{\nu=1}^{n} \omega_\nu(x+k) - \prod_{\nu=1}^{n} \omega_\nu(x+k+1) \right\} \\
&+ 2^n \{(x+K-1) H(x+K-1) \prod_{\nu=1}^{n} \omega_\nu(x+K) - (x-1) H(x-1, B) \prod_{\nu=1}^{n} \omega_\nu(x)\} \\
&\qquad - \frac{(c(B))^2}{4 \cdot 2^n \Omega(B)} \prod_{\nu=1}^{n} (\beta_\nu - \alpha_\nu).
\end{aligned}\right\} (21)$$

Remarquons maintenant que d'après (6) et (5)

$$H(x+k, B) \geqq c(B) \prod_{\nu=1}^{n} (\beta_\nu - \alpha_\nu), \text{ si } x \geqq N_0, \quad \ldots \quad (22)$$

$$0 \leqq H(x, B) \leqq 1 \qquad (x \geqq 1). \quad \ldots\ldots \quad (23)$$

D'après (8) nous tirons de (23)

$$(x+K-1) H(x+K-1) \prod_{\nu=1}^{n} \omega_\nu(x+K) - (x-1) H(x-1, B) \prod_{\nu=1}^{n} \omega_\nu(x) \to 0, \text{ si } x \to \infty.$$

Alors il découle de (21) et (22) pour $x \geqq X_0''$ $(X_0'' \geqq Max(X_0', N_0))$

$$\begin{aligned}
&\sum_{k=0}^{K-1} m\, F_{x,k}(B) \geqq 2^n c(B) \prod_{\nu=1}^{n} (\beta_\nu - \alpha_\nu) \sum_{k=0}^{K-1} \left\{ \prod_{\nu=1}^{n} \omega_\nu(x+k) - \prod_{\nu=1}^{n} \omega_\nu(x+k+1) \right\} (x+k) \\
&- \frac{(c(B))^2}{3 \cdot 2^n \cdot \Omega(B)} \prod_{\nu=1}^{n} (\beta_\nu - \alpha_\nu) = 2^n c(B) \prod_{\nu=1}^{n} (\beta_\nu - \alpha_\nu) \sum_{k=0}^{K-1} \prod_{\nu=1}^{n} \omega_\nu(x+k) - \frac{(c(B))^2}{3 \cdot 2^n \cdot \Omega(B)} \prod_{\nu=1}^{n} (\beta_\nu - \alpha_\nu) + \\
&+ 2^n c(B) \prod_{\nu=1}^{n} (\beta_\nu - \alpha_\nu) \{(x-1) \prod_{\nu=1}^{n} \omega_\nu(x) - (x+K) \prod_{\nu=1}^{n} \omega_\nu(x+K)\}.
\end{aligned}$$

C'est à dire qu'on a pour $x \geqq X_0$ $(X_0 \geqq X_0'')$

$$\sum_{k=0}^{K-1} m F_{x,k}(B) \geqq \frac{2^n (c(B))^2 \prod_{\nu=1}^{n} (\beta_\nu - a_\nu)}{2 \cdot 4^n \cdot \Omega(B)} - \frac{(c(B))^2}{3 \cdot 2^n \cdot \Omega(B)} \prod_{\nu=1}^{n} (\beta_\nu - a_\nu)$$

$$- \frac{(c(B))^2}{12 \cdot 2^n \cdot \Omega(B)} \prod_{\nu=1}^{n} (\beta_\nu - a_\nu) = \frac{(c(B))^2}{12 \cdot 2^n \cdot \Omega(B)} \prod_{\nu=1}^{n} (\beta_\nu - a_\nu)$$

à cause de (20) et de (8). Ça prouve le lemme.

§ 3. *Démonstration des Théorèmes 3 et 4.*

Prenons un point quelconque $A = (a_1, a_2, \ldots, a_n)$ du cube

$$0 < u_\nu < 1 \qquad (\nu = 1, 2, \ldots, n) \quad \ldots \ldots \quad (24)$$

et un parallélépipède arbitraire $\quad a_\nu < u_\nu < \beta_\nu \quad (\nu = 1, 2, \ldots, n) \quad \ldots \quad (25)$

de centre $A$ et appartenant totalement au cube (24).

Remarquons d'abord que sous les conditions des théorèmes 3 et 4 on peut poser dans (18) en tout cas

$$\Omega(B) = \prod_{\nu=1}^{n} \left(1 + \frac{1}{\beta_\nu - a_\nu}\right),$$

car $d_\nu(x+i, x+k)$ désigne un nombre naturel. Sous les conditions du théorème 4 on peut même poser

$$\Omega(B) = 2^n,$$

á cause de la condition $d_\nu(z, x) \to \infty$ pour $z \to \infty$.

Considérons pour $x = 1, 2, \ldots$ les ensembles $E_x(B)$ définis dans le lemme 2 et posons

$$E_x^*(B) = E_x(B) \dotplus E_{x+1}(B) \dotplus \ldots \qquad (x \geqq 1). \quad \ldots \quad (26)$$

Les conditions du lemme 3 étant toutes remplies, (19) entraîne que la mesure de $E_x^*(B)$ est au moins égale à

$$\frac{(c(B))^2}{12 \cdot 2^n \cdot \Omega(B)} \prod_{\nu=1}^{n} (\beta_\nu - a_\nu); \quad \ldots \ldots \quad (27)$$

comme $E_x^*(B) \supset E_{x+1}^*(B)$ la mesure du produit $E^*(B)$ des ensembles $E_x^*(B)$ est donc aussi au moins égal à (27). À cause de la définition de $E_x(B)$ nous concluons que la densité de $E^*(B)$ par rapport au parallélépipède (25) est supérieure à

$$\frac{(c(B))^2}{24 \cdot 2^n \cdot \Omega(B)}.$$

Remarquons maintenant que l'ensemble $G$ des points $(\theta_1, \theta_2, \ldots, \theta_n)$ de (24) admettant une infinité de solutions entières $x \geqq 1, y_1, y_2, \ldots, y_n$ de (4) contient l'ensemble $E^*(B)$. Ainsi le théorème 3 est démontré.

Démontrons maintenant le théorème 4. D'abord je fais remarquer que sous les conditions du théorème 4 le nombre $c$ ne dépend pas du système $B$. En outre nous posons $\Omega(B) = 2^n$. Alors la densité de l'ensemble $G$ par rapport au parallélépipède (25) est supérieure à

$$\frac{c^2}{24 \cdot 4^n}$$

c'est à dire que la densité inférieure de $G$ en $A$ est supérieure à $\frac{c^2}{24 \cdot 4^n}$, le parallélépipède (25) étant arbitraire. Comme $A$ est un point arbitraire du cube (24), ça prouve le théorème d'après une proposition bien connue de la théorie de la mesure.

**Mathematics.** — *Zur projektiven Differentialgeometrie der Regelflächen im $R_4$.* (Neunte Mitteilung.) Von W. J. BOS. (Communicated by Prof. R. WEITZENBÖCK.)

(Communicated at the meeting of January 31, 1942.)

Wenn die einfachste Differentialinvariante $Q$ verschwindet, ist auch $R$ eine Differentialinvariante, wie man leicht aus den Gleichungen (129) und (130) ersieht.

Wir untersuchen hier die geometrische Bedeutung der beiden Fälle $Q \equiv 0$, $R \not\equiv 0$ und $Q \equiv 0$, $R \equiv 0$.

Weiter behandeln wir die Ebenen, welche vier aufeinanderfolgende Erzeugenden schneiden. Wir finden hier zwei Ebenenbüschel, deren Ebenen ausserdem noch den Heftpunkt $H$ enthalten. In jedem Büschel gibt es eine Ebene, die fünf aufeinanderfolgende Erzeugenden schneidet.

§ 26.

Wir betrachten im Folgenden nur allgemeine Regelflächen ($M'_{02} \not\equiv 0$, $H \not\equiv 0$). In der Mitteilung II fanden wir (52): $M'_{02}(\alpha_{ik}) = -16\, Q^2 M'_{02}$.

Die Heftfläche ist also abwickelbar wenn $Q \equiv 0$. Und umgekehrt:

Wenn es eine Heftfläche gibt bei einer allgemeinen Regelfläche, und diese Heftfläche ist abwickelbar, dann ist $Q \equiv 0$.

Mit Hilfe der Gleichungen (84) und (85) finden wir für die Oskulationsebene der Heftkurve im Heftpunkte:

$$h'\pi)^2 = (HH_1H_2\pi^2) = \tfrac{4}{3}\, Q\, (\pi_{02,04} + \pi_{02,22}) + \tfrac{4}{27}\, (R - 6\, Q')\, \pi_{02,03} = 0. \qquad (239)$$

Wenn $Q \equiv 0$ (also auch $Q' \equiv 0$) bekommen wir: $(h'\pi)^2 = \frac{4}{27} R \pi_{02,03} = 0$, d.h.: Die Oskulationsebene der Heftkurve im Heftpunkte ist die Heftebene. Also:

Wenn die Heftfläche abwickelbar ist, dann ist die Heftkurve eine asymptotische Kurve der Fläche; sonst eine quasi-asymptotische Kurve [1]). Die Tangente $h_{ik}$ im Heftpunkte $H$ an die Heftkurve, die Gerade $m_{ik}$, und die Heftgerade $\alpha_{ik}$ fallen jetzt zusammen (Gl. (37) und (57)).

Wenn es keine Heftfläche gibt, d.h. wenn $\alpha_{ik}$ konstant ist, dann ist die Heftkurve eine Gerade. Die Geraden $h_{ik}$, $\alpha_{ik}$ und $m_{ik}$ werden in diesem Falle wieder zusammenfallen; also ist $Q \equiv 0$. Die Oskulationsebene (239) der Heftkurve wird unbestimmt. Da die Heftebene nicht unbestimmt wird, ist also $R \equiv 0$.

Umgekehrt: Wenn $Q \equiv 0$ und $R \equiv 0$, dann ist die Oskulationsebene der Heftkurve unbestimmt. Die Heftkurve kann aber nicht nur ein Punkt sein, denn dann würde dieser Punkt auf allen Erzeugenden liegen, die Fläche also ein Kegel sein. Die Heftkurve ist also eine Gerade.

Im Anschlusse der in der Mitteilung I gegebene Klassifikation haben wir also gefunden:

$M'_{02} \not\equiv 0$, $H \not\equiv 0$, *allgemeine Fläche.*
- $Q \not\equiv 0$, *die Heftfläche ist nicht abwickelbar* (aufeinanderfolgende Heftgeraden schneiden sich im Allgemeinen nicht).
- $Q \equiv 0$, $R \not\equiv 0$, *die Heftfläche ist abwickelbar* (aufeinanderfolgende Heftgeraden schneiden sich immer).
- $Q \equiv 0$, $R \equiv 0$, *die Heftkurve ist eine Gerade* (die Heftgeraden fallen zusammen).

[1]) Vgl. E. BOMPIANI: Rendiconto Palermo XXXVII (1914), 305—331.

Wir bemerken noch: Wenn $Q \equiv 0$ ist, finden wir für den Oskulationsraum der Heftkurve im Heftpunkte (86):

$$(H'x) \equiv (HH_1 H_2 H_3 x) = -\tfrac{1}{3} x_{02} (H_2)_{22} (H_3)_{03} = 0.$$

Mit (85) und (66) haben wir also:

$$(H'x) = -\tfrac{1}{3} (\tfrac{4}{9} R - \tfrac{8}{3} Q') (\tfrac{1}{3} R - 6 Q') x_{02} = -\tfrac{4}{81} R^2 x_{02} = 0. \quad (240)$$

Die Heftkurve liegt in einem $R_3$ wenn

$$\triangle_H = (HH_1 H_2 H_3 H_4) = (H' H_4) = 0.$$

Im Falle $Q \equiv 0$ gibt dies mit (88) und (240):

$$\triangle_H = -\tfrac{4}{81} R^2 (H_4)_{02}.$$

Eine einfache Berechnung zeigt $(H_4)_{02} = -\frac{2}{9} R + \frac{20}{3} Q'$. Also wird:

$$\triangle_H = \tfrac{8}{729} R^3. \quad (241)$$

Hieraus ersieht man:

Die Heftkurve einer Regelfläche, deren Heftfläche abwickelbar ist, liegt nicht in einem $R_3$ und ist also auch keine ebene Kurve.

§ 27.

Den Ebenen, die drei aufeinanderfolgende Erzeugenden schneiden, begegneten wir in der 8. Mitteilung. Wir fanden dort (238):

Der quadratische Linienkomplex $(01^2 \pi^2)(02^2 u^2) \equiv 0_{1\pi, 2\pi} = 0$ ist der Ort jener Geraden wodurch sich Ebenen legen lassen, die drei aufeinanderfolgende Erzeugenden schneiden. Die $\infty^3$ Ebenen kann man auch wie folgt erhalten:

Der Raum $v'$ schneidet die Fläche $a_{ik}(t)$ in der Kurve:

$$C_{v'} \equiv a_{v'} a_{u'} \equiv 0_{v'} 0_{u'} + t \,.\, 1_{v'} 1_{u'} + \tfrac{1}{2} t^2 \,.\, 2_{v'} 2_{u'} + \ldots \quad (242)$$

Die Gleichung der Oskulationsebene der Kurve $C_{v'}$ im Schnittpunkte von $v'$ mit $a_{ik}$ (0) lautet:

$$(012\, \pi^2)\, 0_{v'} 1_{v'} 2_{v'} \equiv -(0^2 12 \pi)\, 1_{v'} 2_{v'} \,.\, \pi_{v'} = 0. \quad (243)$$

Die Oskulationsebene erscheint also als Schnittebene der Räume $(v'x) = 0$ und $(w'x) \equiv (0^2 12\, x)\, 1_{v'} 2_{v'} = 0$. Die Ebenen (243) sind die Ebenen, welche drei aufeinanderfolgende Erzeugenden schneiden.

Es gibt $\infty^2$ Ebenen, welche vier Geraden $a$, $b$, $c$ und $d$ allgemeiner Lage schneiden; diese Ebenen kann man auf folgende Weise erhalten. Man wählt einen Punkt $P$ auf $a$, und einen Punkt $Q$ auf $b$. Dann gibt es eine Gerade $e$, die $PQ$, $c$ und $d$ schneidet. Die Ebene der Geraden $PQ$ und $e$ schneidet die vier Geraden $a$, $b$, $c$ und $d$.

Wir suchen jetzt die Ebenen, welche vier aufeinanderfolgende Erzeugenden einer Regelfläche schneiden.

Die Oskulationsebene (243) der Kurve $C_{v'}$ (242) hyperoskuliert wenn

$$(0123\, x)\, 0_{v'} 1_{v'} 2_{v'} 3_{v'} \equiv (0^2 123)\, 1_{v'} 2_{v'} 3_{v'} \,.\, x_{v'} \equiv 0 \, \{x\}.$$

Die Differentialkontravariante

$$K = (0^2 123)\, 1_{u'} 2_{u'} 3_{u'} = 0 \quad (244)$$

gibt also die Räume $u'$, die vier aufeinanderfolgende Erzeugenden in koplanairen Punkten schneiden.

Oder: $K = 0$ *ist die Gleichung in Raumkoordinaten der Mannigfaltigkeit der* $\infty^2$ *Ebenen, die vier aufeinanderfolgende Erzeugenden schneiden.*

Bei einer Fläche $F_2^3$ dritten Grades wird eine Ebene, welche vier Erzeugenden schneidet, die Fläche in einer ebenen Kurve schneiden. Tatsächlich fanden wir in der sechsten Mitteilung, dass $K = 0$ die Gleichung der $F_2^3$ in Raumkoordinaten darstellt (211*a*). Die $\infty^2$ Ebenen sind in diesem Falle die Ebenen der $\infty^2$ Kegelschnitte auf $F_2^3$.

Betrachten wir wieder die vier Geraden *a*, *b*, *c* und *d* allgemeiner Lage im $R_4$. Sei *e* die Transversale der Geraden *a*, *b* und *c*; *P* der Schnittpunkt von *b* mit *e*; und *Q* der Schnittpunkt der Gerade *d* mit dem Raume durch *a* und *c*; dann sieht man geometrisch leicht ein, dass die Ebenen durch *P*, welche die vier Geraden schneiden, zwei Ebenenbüschel bilden. Das erste Ebenenbüschel (*A*) liegt im Raume durch *e* und *d* und hat die Achse *e*. Das zweite Büschel (*B*) liegt im Raume durch *a* und *c* und hat die Achse *PQ*.

Sind *a*, *b*, *c* und *d* vier Erzeugende einer Regelfläche *F*, dann können wir nach der Grenzlage der Ebenenbüschel *A* und *B* fragen, wenn die vier Erzeugenden gegen eine bestimmte Erzeugende $a_{ik}(0)$ konvergieren.

Wir denken wieder $Q \neq 0$ und behaupten:

*Es gibt zwei Ebenenbüschel, deren Achsen den Heftpunkt H enthalten und deren Ebenen vier aufeinanderfolgende Erzeugende schneiden.* Wir nennen die Ebenen dieser Ebenenbüschel: *„die Vierpunktebenen A und B".*

Die Vierpunktebenen *A* sind die Ebenen im Tangentialraume $x_{02} = 0$, durch die Verbindungsgerade der Punkte $H = 0_{22}\, 0_{u'} = 0$ und $3_{02}\, 3_{u'} = 0$, d.h. durch die Gerade:

$$\left.\begin{aligned} 0_{22}\, 3_{02}(03)_{ik} &= +\, 0_{22}\, \dot{0}_{23}\, (\dot{0}0) + 0_{22}\, 2_{03}\, (20) = \\ &= -\tfrac{1}{2}\, 0_{22,23} \,.\, 0_{ik} - a_{ik} = \tfrac{2}{3}\, Q \,.\, 0_{ik} - a_{ik} \end{aligned}\right\} \quad . \quad . \quad . \quad (245)$$

Die Vierpunktebenen *B* sind die Ebenen durch die Heftgerade $a_{ik}$ im Raume $(a^2\, 3^2\, x) = 0$, d.h. im Raume $(B'\, x) \equiv R\, x_{02} - 2Q\, x_{03} = 0$ (Vgl. (98)). Denn wir haben:

$$(a^2\, 3^2\, x) = 0_{22}\, 2_{03}\, (023^2\, x) = 0_{22}\, (203^2\, x)\, 2_{03} = 0_{22}\, (2^2\, 03\, x)\, 3_{03} - \tfrac{1}{2} \,.\, 0_{22,23} \,.\, x_{03} =$$
$$= -\tfrac{1}{2} \,.\, 0_{22}\, \dot{0}_{33}\, (\dot{0}02^2\, x) + \tfrac{2}{3}\, Q \,.\, x_{03} = \tfrac{1}{4} \,.\, 0_{22,33} \,.\, x_{02} + \tfrac{2}{3}\, Q \,.\, x_{03}.$$

Also:

$$(a^2\, 3^2\, x) = -\tfrac{1}{3}(R \,.\, x_{02} - 2\, Q \,.\, x_{03}) = -\tfrac{1}{3}(B'\, x). \quad . \quad . \quad . \quad (246)$$

*Beweis:* Die Vierpunktebenen *A* können wir darstellen durch:

$$A_{ikl} = \tfrac{2}{3}\, Q \,.\, (0^2\, y)_{ikl} - (a^2\, y)_{ikl} \quad \text{mit} \quad y_{02} = 0.$$

Wenn $a_{ik}(t)$ wieder die Erzeugende von *F* ist, dann haben wir zu beweisen, dass der Ausdruck $(A^3\, a^2)$ mit Termen in $t^4$ anfängt; also ist zu beweisen:

$$(A^3\, 0^2) = 0, \ (A^3\, 1^2) = 0, \ (A^3\, 2^2) = 0, \ (A^3\, 3^2) = 0.$$

In der Tat haben wir:

$$(A^3\, 0^2) = \tfrac{2}{3}\, Q \,.\, (0^2\, 0^2\, y) - (a^2\, 0^2\, y) = 0, \ \text{denn} \ M'_{00} = 0 \ \text{und} \ M'_{\alpha 0} = 0.$$

$$(A^3\, 1^2) = \tfrac{2}{3}\, Q \,.\, (0^2\, 1^2\, y) - (a^2\, 1^2\, y) = 0, \ \text{denn} \ M'_{01} = 0 \ \text{und} \ M'_{\alpha 1} = 0.$$

$$(A^3\, 2^2) = \tfrac{2}{3}\, Q \,.\, (0^2\, 2^2\, y) - (a^2\, 2^2\, y) = 0, \ \text{denn} \ y_{02} = 0 \ \text{und}$$
$$(a^2\, 2^2\, x) = -\, 2_{03}\, (022^2\, x)\, 0_{22} = +\, \tfrac{1}{2}\, 2_{03,02} \,.\, x_{22} = 0$$

$$(A^3\, 3^2) = \tfrac{2}{3}\, Q \,.\, (0^2\, 3^2\, y) - (a^2\, 3^2\, y) = \tfrac{2}{3}\, Q \,.\, y_{03} + \tfrac{1}{3} R \,.\, y_{02} - \tfrac{2}{3}\, Q \,.\, y_{03} = \tfrac{1}{3} R \,.\, y_{02} = 0$$

(vgl. (246)).

Für die Vierpunktebenen $B$ haben wir die Darstellung:

$$B_{ikl} = (\alpha^2 y)_{ikl} \text{ mit } R \cdot y_{02} - 2\,Q \cdot y_{03} = 0.$$

Wir haben also zu beweisen:

$$(B^3\,0^2) = 0, \ (B^3\,1^2) = 0, \ (B^3\,2^2) = 0, \ (B^3\,3^2) = 0.$$

Wir finden tatsächlich:

$$(B^3\,0^2) = (\alpha^2\,0^2\,y) = 0, (B^3\,1^2) = (\alpha^2\,1^2\,y) = 0, (B^3\,2^2) = (\alpha^2\,2^2\,y) = 0 \text{ und mit } (246):$$

$$(B^3\,3^2) = (\alpha^2\,3^2\,y) = -\tfrac{1}{3}(R \cdot y_{02} - 2\,Q \cdot y_{03}) = 0.$$

Die Vierpunktebenen $A$ und $B$ haben eine Ebene gemeinsam: die Heftebene $\pi_{02,03} = 0$.

Wenn $G$ wieder den Schnittpunkt der Beigerade $g_{ik}$ (93) mit der Heftebene darstellt, dann zeigt die Gleichung (95), dass die Achse (245) der Vierpunktebenen $A$ die Gerade $HG$ ist.

Die Vierpunktebenen $B$ liegen im Raume $(B'\,x) = 0$, und diesen Raum fanden wir in der Mitteilung III durch Verbindung der Geraden $g_{ik}$ und $0_{ik}$. Den zwei letzten Bemerkungen entnimmt man:

*Die Vierpunktebenen $A$ und $B$ schneiden die Beigerade $g_{ik}$.*

Dies war auch geometrisch zu erwarten.

## § 28.

Der Tangentialraum $x_{02} = 0$ schneidet die Fläche $F$ in zwei zusammenfallenden Erzeugenden und in einer Kurve durch den Heftpunkt.

Die Ebenen im Tangentialraume durch die Tangente dieser Kurve im Heftpunkte $H$ sind die Vierpunktebenen $A$.

Es ist deutlich, dass die Oskulationsebene dieser Kurve im Punkte $H$ fünf aufeinanderfolgende Erzeugenden schneidet. Also:

*Die Vierpunktebenen $A$ enthalten eine Fünfpunktebene $A$.*

Die Gleichung dieser Ebene bekommen wir wie folgt:

Der Punkt $y$ im Raume $x_{02} = 0$ soll derartig gewählt werden, dass jetzt auch $(A^3\,4^2) = \frac{2}{3}\,Q \cdot y_{04} - (\alpha^2\,4^2\,y) = 0$ wird. Der Punkt $y$ muss folglich in der Schnittebene der zwei Räume $x_{02} = 0$ und $\frac{2}{3}\,Q \cdot x_{04} - (\alpha^2\,4^2\,x) = 0$ liegen. Die gesuchte Ebene ist also die Schnittebene dieser Räume. Für den Ausdruck $(\alpha^2\,4^2\,x)$ finden wir:

$$(\alpha^2\,4^2\,x) = 0_{22}\,2_{03}\,(024^2\,x) = 0_{22}\,(204^2\,x)\,2_{03} = 0_{22}\,(2^2\,04\,x)\,4_{03} - \tfrac{1}{2} \cdot 0_{22,24} \cdot x_{03}.$$

Der erste Term gibt:

$$0_{22}\,(2^2\,04\,x)\,4_{03} = \tfrac{1}{2} \cdot 4_{03,22} \cdot x_{02} - \tfrac{1}{2} \cdot 4_{03,02} \cdot x_{22} = \tfrac{1}{2}\,4_{03,22} \cdot x_{02} + \tfrac{1}{2} \cdot 2_{03,04} \cdot x_{22} =$$

$$= -\,2_{03,24} \cdot x_{02} - 2 \cdot 2_{03,13} \cdot x_{22}.$$

Also nach (110):

$$0_{22}\,(2^2\,04\,x)\,4_{03} = -\,(\tfrac{1}{3}\,R' + S)\,x_{02} - 2 \cdot Q \cdot x_{22}.$$

Der zweite Term wird wegen (1) und (104):

$$-\,\tfrac{1}{2}\,0_{22,24} \cdot x_{03} = \tfrac{2}{3}\,0_{13,24} \cdot x_{03} = (-\,\tfrac{5}{9}\,R + \tfrac{2}{3}\,Q')\,x_{03}.$$

Damit wird:

$$(\alpha^2\,4^2\,x) = -(\tfrac{1}{3}\,R' + S)\,x_{02} - (\tfrac{5}{9}\,R - \tfrac{2}{3}\,Q')\,x_{03} - 2 \cdot Q \cdot x_{22}. \qquad (247)$$

Die Gleichung der Fünfpunktebene $A$ wird also:

$$\tfrac{2}{3}\, Q \,.\, \pi_{02,04} - \pi_{02,\alpha 4} = (\tfrac{5}{9}\, R - \tfrac{2}{3}\, Q')\, \pi_{02,03} + 2 \,.\, Q \,.\, \pi_{02,22} + \tfrac{2}{3}\, Q \,.\, \pi_{02,04} = 0.$$

Oder:

$$(A'\,\pi)^2 = (5\,R - 6\,Q')\,\pi_{02,03} + 18\,Q \,.\, \pi_{02,22} + 6\,Q \,.\, \pi_{02,04} = 0. \qquad (248)$$

Auf dieselbe Weise erhalten wir:

*Die Vierpunktebenen $B$ enthalten eine Fünfpunktebene $B$.*

Diese Ebene ist die Schnittebene der Räume $R \,.\, x_{02} - 2 \,.\, Q \,.\, x_{03} = 0$ und $(\alpha^2\, 4^2\, x) = 0$. Die Gleichung dieser Ebene lautet also:

$$R \,.\, \pi_{02,\alpha 4} - 2 \,.\, Q \,.\, \pi_{03,\alpha 4} = (-\tfrac{5}{9}\, R^2 + \tfrac{2}{3}\, R \,.\, Q' - \tfrac{2}{3}\, Q \,.\, R' - 2 \,.\, Q \,.\, S)\, \pi_{02,03} - \\ - 2\, R\, Q\, \pi_{02,22} + 4\, Q^2 \,.\, \pi_{03,22} = 0.$$

Oder:

$$\left.\begin{aligned} (B'\,\pi)^2 = (5\,R^2 - 6\,R \,.\, Q' + 6\,Q \,.\, R' + 18\,Q \,.\, S)\,\pi_{02,03} + \\ + 18\,R \,.\, Q \,.\, \pi_{02,22} - 36\,Q^2 \,.\, \pi_{03,22} = 0 \end{aligned}\right\} \,. \qquad (249)$$

**Anatomy.** — *Enkele beschouwingen naar aanleiding van de onderzoekingen van* VISSER[1]. Door TH. E. DE JONGE. (Communicated by Prof. M. W. WOERDEMAN.)

(Communicated at the meeting of January 31, 1942.)

VISSER, die ons in een gedegen studie een volledig overzicht geeft van de wortelvergroeiingen bij de bovenkaaksmolares van 's menschen gebit, licht zijne mededeeling toe met een aantal cijfers, waarvan wij de voor onze beschouwingen belangrijkste in onderstaande tabel samenvatten.

| Aantal onderzochte molares | Drie divergeerende wortels | Vergroeiïng van beide buccale wortels | Vergroeiïng van palatinalen met distobuccalen wortel | Vergroeiïng van palatinalen met mesiobuccalen wortel |
|---|---|---|---|---|
| 135 m. I | 74 of 55% | — | 61 of 45 % | — |
| 93 m. II | 23 of 25% | — | 70 of 75 % | — |
| 2867 M. I | 2568 of 90% | 78 of 3 % | 206 of 7.2% | 3 of 0.3% |
| 2859 M. II | 1559 of 55% | 398 of 14 % | 37 of 1.3% | 429 of 15 % |
| 2431 M. III | 686 of 28% | 281 of 11.5% | 267 of 11 % | 147 of 6 % |

Terecht vestigt de schrijver er de aandacht op, dat deze cijfers belangrijk afwijken van die van vroegere onderzoekers en ter verklaring daarvan legt hij den nadruk op het omvangrijke materiaal, dat hem ten dienste stond[2]: daardoor toch — en mede doordien aldus de fout der persoonlijke waarneming zóó gering wordt, dat zij practisch verwaarloosd kan worden — winnen de door hem gevonden uitkomsten aanmerkelijk aan waarde.

Daarnaast echter vormen, gelijk bij de verschillende kroonformaties, ook rasverschillen eenen factor van niet te onderschatten beteekenis. Zoo kenmerken zich b.v. de molaarkronen der recente Hollandsche bevolking door eene uitgesproken vereenvoudigingstendentie: hetzelfde geldt uitteraard voor hare wortels.

En nu moge ons de schrijver op klare wijze de vraag beantwoord hebben, *hoe* zich hunne structuurvereenvoudiging aan ons oog voordoet, van den achtergrond dezer vraag dringt zich onverbiddelijk eene tweede vraag naar voren: *waarom*?

Waarom deze verschillende wortelstructuren? Aan de beantwoording dezer vraag ga eene korte beschouwing vooraf betreffende de vormontwikkeling van 's menschen gebit.

Hoe verschillend de talrijke onderzoekers, die zich met dit vraagstuk beziggehouden hebben, ook denken mogen omtrent de wijze, waarop zich dit uit primitiever vormen ontwikkeld heeft, algemeen onderscheidt men niettemin twee stadia in zijne phylogenese; eerst eene morphologisch-progressieve phase, welke bij onze bovenmolares culmineert in wat wij ook thans nog als prototype van den normalen molaarvorm beschouwen kunnen: eene kroon, opgebouwd uit vier knobbels, waarnaast zich veelal nog als vijfde element het mesiolinguale tuberculum CARABELLI manifesteert.

Dan echter maakt onze geheele gebitsstructuur eene morphologisch-regressieve ontwikkelingsphase door. Het duidelijkst zien wij deze vereenvoudigingstendentie bij den tweeden molaris, die in meer dan de helft der gevallen reeds drieknobbelig is, terwijl normaliter ook het tuberculum CARABELLI niet of nauwelijks meer tot ontwikkeling komt.

Dat een dergelijke structuurmodificatie — die bovendien veelal gepaard gaat met een vrij sterke anterodistale afplatting der kroon — niet zonder invloed kan blijven op de wortelformatie, ligt voor de hand.

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1) *Tijdschrift voor Tandheelkunde,* Januari 1942.

2) Verzameling van het Ontleedkundig Laboratorium te Amsterdam.

In feite toch is de wortel niets anders dan een steunapparaat van de kroon, hetwelk in de vormontwikkeling van deze laatste de noodzakelijke voorwaarde voor eigene differentiatie vindt. Zoo zien wij b.v. hoe de primitieve kegelvorm van den snijtandswortel allengs plaats maakt voor het dimere type, dat wij bij de præmolares kennen en welks buccale en linguale zône in zekeren zin de voortzetting vormen der beide kroonknobbels. Nu kunnen deze segmenten uitgroeien tot twee wortels doch verder voortschrijdende molarisatie der kroon gaat bovendien gepaard met anterodistale differentiatie van den wortel, die tenslotte haar hoogtepunt bereikt in de drie gespreide radices van den eersten molaris.

Zoo kan het derhalve moeilijk anders of óók de regressieve ontwikkelingsphase drukt in gelijke mate haren stempel op de configuratie der wortels. Het duidelijkst komt dit wel bij den tweeden molaris tot uitdrukking, want deze mag, véél meer dan zijn distale synergeet, als *het* classieke voorbeeld van structuurvereenvoudiging gelden. Zooals nu de reductie van zijnen achtersten lingualen kroonknobbel haren weerslag vindt in de versmelting van palatinalen met voorsten buccalen wortel, zoo hebben wij in nog verdergaande coalescentie der wortels onderling — welke tenslotte in een kegelvorm culmineert — de resultante te zien van nòg progrediënter kroonvereenvoudiging.

Ook de wortels der beide andere molares zullen m.m. den invloed der bovenbeschreven structuurvereenvoudiging ondergaan. Twee bijzonderheden nochtans vragen de aandacht. Vooreerst: de vereenvoudigingstendentie draagt bij den derden molaris — en in nog véél hooger mate geldt zulks voor den eersten molaris — een veel minder geprononceerd karakter; ook in de verhoudingscijfers hunner verschillende wortelstructuren vinden wij dit onderscheid op sprekende wijze geregistreerd.

Een tweede, veel prægnanter verschilpunt betreft in het bijzonder den eersten molaris: terwijl bij den tweeden immers versmelting van den palatinalen met den *voorsten* buccalen wortel domineert, blijkt hier de palatinale wortel zich nagenoeg altijd met den *achtersten* buccalen te vereenigen. Deze tegenstelling is te opmerkelijker, wijl de vereenvoudiging van beider kronen, hoezeer gradueel verschillend, principieel eenzelfde karakter draagt. En daar het mede op grond van statische en dynamische factoren zoo al niet onaannemelijk dan toch in ieder geval uiterst onwaarschijnlijk geacht moet worden, dat de vereenvoudiging der wortelstructuur zich bij den eersten molaris op andere wijze voltrekken zoude dan bij den tweeden, ligt de vraag voor de hand, hoe deze controverse te verklaren.

De beantwoording dezer vraag stelt wel op duidelijke wijze in het licht, hoe gelukkige gedachte het was, dat VISSER óók de melkmolares in zijn onderzoek betrokken heeft. Vergelijking toch der verschillende cijfergroepen toont ons, dat van versmelting van palatinale met mesiobuccale radix — die bij den tweeden blijvenden molaris in zekeren zin het morphologisch complement vormt van de reductie van zijne distobuccale kroon-cuspis en die VISSER bij den eersten blijvenden molaris in het geheel slechts driemalen telde — bij de melkmolares evenmin sprake is als van coalescentie der beide buccale wortels onderling.

Anderszijds: vergroeiïng van den palatinalen met den distobuccalen wortel blijkt bij de melkmolares *ondanks de buitengewoon sterke divergentie hunner wortels* een nog aanmerkelijk hooger percentage van gevallen te omvatten dan bij den voorsten blijvenden molaris het geval is.

Wanneer wij daarnaast in aanmerking nemen, dat de melkmolares, wel verre van onderhevig te zijn aan retrogressieve vorminvloeden, véél zuiverder dan de blijvende elementen hun oorspronkelijk morphologisch karakter hebben weten te bewaren — dit geldt voor de structuur hunner kronen, in gelijke mate derhalve voor hunne wortelformatie — dan luidt onze conclusie aldus: vergroeiïng van den palatinalen met den distobuccalen wortel, die bij de melkmolares immers onmogelijk de uitdrukking kan zijn eener vereenvoudigingstendentie, behoort veeleer tot die primitieve kenmerken, die zich in de lacteale dentitie zooveel langer en zooveel zuiverder hebben weten te handhaven dan in de blijvende reeks. Even verklaarbaar onder dezen zelfden gezichtshoek is het ten eenenmale achterwege blijven van vergroeiïng tusschen palatinalen en mesiobuccalen wortel!

Tot zooverre de melkmolares. Niet anders is het ons inziens met den eersten blijvenden

molaris gesteld, waar feiten en cijfers de bovengegeven voorstelling van zaken al evenzeer schijnen te bevestigen.

Immers, wel blijken bij dezen de eerste symptomen eener beginnende structuurvereenvoudiging aanwezig, van een bepaalden invloed op zijn wortel kan echter nauwelijks nog sprake zijn. Dat VISSER derhalve vergroeiïng van de palatinale met de mesiobuccale radix in slechts drie gevallen waarnam, behoeft ons geenszins te verrassen!

En wat de versmelting met den distobuccalen wortel betreft, reeds tevoren wezen wij erop, dat de verklaring ervan als vereenvoudigingsverschijnsel moeilijk in overeenstemming te brengen is met de wijze, waarop zich deze bij den tweeden molaris voordoet. Trouwens, ook de betrekkelijk hooge frequentie dezer vergroeiïng bij den eersten molaris verzet zich tegen deze interpretatie.

Beschouwen wij haar daarentegen, gelijk bij de melkmolares, als een vorm, die de herinnering bewaart aan eene vroegere phase in de ontwikkeling der wortelstructuur, dan worden niet slechts bovengenoemde bezwaren ontzenuwd doch vinden wij tevens opnieuw de genetische relatie tusschen melkmolares en eersten blijvenden molaris op marquante wijze bevestigd!


*Samenvatting.*

De vereenvoudiging der wortelformatie maakt zich bij de bovenkaaksmolares der blijvende reeks in eerste instantie kenbaar door versmelting van palatinale met mesiobuccale radix.

Daarnaast echter kan — bij voorkeur bij de beide melkmolares, doch in mindere mate ook bij den eersten blijvenden molaris — de ontwikkeling van een beensæptum of beenlijst eene verbinding van palatinalen met distobuccalen wortel tot stand brengen, die de herinnering aan een vroeger stadium in de vormgenese der wortelstructuur gefixeerd houdt.

*Zusammenfassung.*

Die Vereinfachung der Wurzelformation manifestiert sich bei den Molaren im Oberkiefer des bleibenden Gebisses an erster Stelle durch Verschmelzung von palatinaler mit mesiobukkaler Radix.

Daneben aber kann — vorzugsweise bei den beiden Milchmolaren, doch auch, obwohl in geringerem Masze, beim ersten bleibenden Molar — die Entwicklung eines Knochensæptums zu einer Verbindung von palatinaler mit distobukkaler Wurzel führen, die die Erinnerung an ein früheres Stadium in der Formgenese der Wurzelstruktur lebendig erhält.

*Summary.*

The simplification of the rootformation is manifested in the permanent molars of the upper jaw in the first place by fusion of the palatinal with the mesiobuccal root.

Besides, the development of an osseous septum can cause the union of the palatinal with the distobuccal root, especially in both the milk molars but, although in a smaller scale, also in the first permanent molar. This union must be seen as the remembrance of a previous stage in the morphogenesis of root structure.

*Résumé.*

La simplification de la formation des racines se manifeste en premier lieu dans les molaires supérieures de la dentition permanente par la fusion des racines palatales et mesiobuccales.

Cependant à côté de cela — et de préférence dans les deux molaires de lait mais aussi, quoique dans une moindre mesure, dans la première molaire permanente — le développement d'un sæptum osseux crée une union de la racine palatale avec la racine distobuccale, qui maintient le souvenir d'une phase antérieure dans la morphogénèse de la structure de la racine.

**Psychologie.** — *Das Problem des Ursprungs der Sprache.* II. Von G. RÉVÉSZ. (Communicated by Prof. A. P. H. A. DE KLEYN.)

(Communicated at the meeting of January 31, 1942.)

## 3. *Ursprungstheorien.*

### A. *Die Ausdruckstheorie.*

Die mimischen und pantomimischen *Ausdrucksbewegungen* sind im Grunde genommen nichts anderes als unmittelbare zwangsläufige Folgeerscheinungen (Reaktionen) innerer Erregtheitszustände. Triebhaft-affektive Zustände lösen die Ausdrucksbewegungen der Freude, Furcht, Abneigung, Zuneigung, des Zorns, Ekels, des Aktivitäts- und Ruhebedürfnisses gleichsam reflektorisch aus. Diese affektiven Zustände bilden mit den entsprechenden Ausdrucksbewegungen sowohl vom biologischen als auch vom psychologischen Standpunkt aus eine *Einheit*: der Affekt und seine Aeusserung, die innere Spannung und ihre Entladung, sind in ein und demselben zeitlich-untrennbaren Akt gegeben [2]). Sie stellen zwei unterschiedbare Manifestationen *desselben Lebensprozesses* dar.

Die Ausdrucksbewegungen als äusseres Zeichen der Gemütsbewegungen haben mit der Sprache nichts Gemeinsames; sie stellen keine Mitteilungsform dar, setzen keinen sozialen Kontakt voraus, werden nicht mit der Absicht vollzogen, eine Verständigung zwischen artgleichen oder artungleichen Wesen herbeizuführen. Das Entscheidende ist, dass die Ausdrucksbewegungen nicht dem Bedürfnis eines gegenseitigen Kontaktes entstammen, ihre Existenz nicht einer Tendenz zu verdanken haben, die jedwede Sprachform beherrscht [3]). Daraus erklärt sich ungezwungen, dass den Ausdrucksbewegungen alle Funktionen der Sprache fehlen. Sie liegen in der Sphäre des Affekt- und Trieblebens; ihnen ist keine geistige Bedeutung eigen. Dem widerspricht auch nicht, dass Ausdrucksbewegungen die Sprache begleiten und unterstützen.

Hat man sich einmal die ursprüngliche Natur der Ausdrucksbewegungen deutlich gemacht, dann wird man nicht mehr in diesen spezifischen Aeusserungen der Affektentladungen die Vorstufe der Sprache erblicken. Die beiden sind vermutlich deshalb miteinander in Verbindung gebracht worden, weil man die unwillkürlichen Ausdrucksbewegungen genetisch zu den willkürlichen in enge Beziehung setzte, kurzum die Ausdrucksbewegungen mit Gebärden identifizierte. Diese Identifizierung ist aber unstatthaft [4]). Die Gebärde ist ein *Verständigungsmittel,* das bereits *sprachbezogen* ist. Gebärden, hinweisenden und deutenden Gesten, liegt bereits die Sprachfunktion zu Grunde. Tiere und ebenso kleine Kinder vor der Periode der Sprachtätigkeit führen keine Gebärden aus; die ersteren nicht, weil ihnen die Sprachfunktion gänzlich fehlt, und die letzteren nicht, weil die Sprachfunktion infolge ihrer geistigen Unreife bei ihnen noch nicht in Wirksamkeit getreten ist. Gebärdensprache ist eine Form der Sprache, die sich mit der Lautsprache in Wechselwirkung entwickelt und auf die Formen der Lautsprache andauernd bestimmend einwirkt. Die Gebärde ist mit dem Wort derart verknüpft, dass sie geradezu einen Teil von ihm zu bilden scheint [5]).

Zusammenfassend können wir sagen, dass die Ableitung der Sprache aus den *Ausdrucksbewegungen* darum unrichtig ist, weil es sich um den Versuch einer Ableitung der Sprache

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[2]) E. CASSIRER, Philosophie der symbolischen Formen, 1923, S. 125.

[3]) Siehe darüber ausführlich im Abschnitt 7.

[4]) Vergl. dazu Abschnitt 3. Punkt D.

[5]) L. LÉVY-BRUHL, Les fonctions mentales dans les sociétés inférieures, 1922; ferner E. CASSIRER, Symbolische Formen, S. 130.

aus Tätigkeiten handelt, die weder konstitutive Merkmale der Sprache enthalten noch von einer gemeinsamen Grundtendenz beherrscht werden; die Ableitung der Sprache aus den *Gebärden* ist aber darum unhaltbar [6]), weil sie auf Funktionen zurückgreift, die bereits eine Art der Sprache darstellen und zu der Lautsprache in engster Beziehung stehen. Der Umstand, dass die Gebärdensprache einige Zeichen verwendet, die aus dem Inventar der ursprünglichen Ausdrucksbewegungen stammen, (wie es z.B. der Fall ist, wenn die Bedeutung „weg" durch Abwendung des Kopfes oder durch eine energische Bewegung der Hand ausgedrückt wird), spricht ebensowenig für die Identität der beiden Funktionen, wie die Tatsache, dass in die Wortsprache Interjektionen (ach, oh) aufgenommen sind, den Ursprung der Sprache aus solchen affektiven Lautäusserungen beweist. Damit kommen wir zu der Interjektionstheorie.

B. *Interjektionstheorie.*

Die *Interjektionstheorie* muss aus denselben Gründen abgelehnt werden wie die Ausdruckstheorie. Die spontanen Lautäusserungen stellen ebenso unwillkürliche Ausdrucksformen von affektiven Zuständen dar wie die Ausdrucksbewegungen der Glieder. Alle Lebewesen, die über Stimmwerkzeuge verfügen, geben Laute von sich. Vielfach sind die emotionalen Lautäusserungen bei Tieren (Vögeln) beinahe die einzig wahrnehmbaren Ausdruckserscheinungen. Bezeichnend für ihre *Ur*sprünglichkeit und *Un*abhängigkeit von der Sprache ist es, dass sie bei noch nicht sprechenden kleinen Kindern im wesentlichen in gleicher Weise in Erscheinung treten wie bei sprechenden Menschen. Sie behalten ihre ursprüngliche Form trotz der geistigen Entwicklung des Menschen bei, da wir für affektive Zustände im eigentlichen Sinne kein sprachliches Ausdrucksmittel (Worte) zu Verfügung haben [7]). Nur durch besondere *Betonung* der Wörter (z.B. Lass mich in *Ruhe*!), also wiederum nur mit Hilfe von den Interjektionen verwandten emotionalen Wortlauten können sie einigermassen ausgedrückt werden.

Wie sollte die *Lautsprache* aus einer Ausdrucksform haben entstehen können, die in der Sprache kein Aequivalent hat und die trotz der Sprache ihre ursprüngliche Form und Funktion unverändert beibehält? Der Laut an sich ist kein die Sprache hervorbringendes und allein von sich aus ihren Charakter bestimmendes Moment. Der Laut kann nur durch den *Sprachsinn* zu Sprachelement umgewandelt werden, genau so wie die Bewegung zur Gebärde. Einzig in Wechselwirkung mit dem Sprachsinn, mit dem Geist, vermag der Laut Bedeutung zu gewinnen und sprachschaffend zu wirken. Wenn die Laute auf Objekte der inneren und äusseren Welt, auf Vorstellungen, Gedanken, bezogen werden, wenn sie aus dem sinnlich-affektiven Zustand heraus und durch den Prozess der Artikulation, Gliederung, Ordnung hindurch gegangen und so Ausdruck des geistigen Bewusstseins geworden sind, erst dann werden die Laute zu Worten, und damit *entsteht* auch erst die Sprache. Laute mit oder ohne begriffliche Bedeutung sind *nicht vergleichbare Gebilde*. Die ersteren sind Produkte des Geistes, des Denkens, des bewussten Formens; die letzteren stellen Aeusserungen des affektiven Zustandes der lebenden Wesen dar.

C. *Nachahmungstheorie.*

Die Erklärung der Lautsprache durch eine Anzahl von onomatopoetischen Lauten stösst auf ähnliche Schwierigkeiten wie die Ausdruckstheorie. STEINTHAL, der bekannteste Vertreter dieser Auffassung, stellt sich den Ursprung der artikulierten Sprachlaute und der inneren Sprachform vermittels der Annahme vor, dass beim Urmenschen, mit jeder besonderen Wahrnehmung eine besondere Artikulation „reflektorisch" [8]) verknüpft

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6) W. WUNDT, Völkerpsychologie, 1911. Band Sprache. Teil I, S. 143 ff.

7) H. MAIER, Psychologie des emotionalen Denkens, 1908, S. 438.

8) STEINTHAL verwendet hier das Wort reflektorisch an Stelle von instinktiv; darum wird seine Anschauung „Reflextheorie" genannt.

war, und zwar eine solche, welche onomatopoetisch war, d. h. mit der zugehörigen Anschauung eine deutliche Aehnlichkeit besass [9])."

Diese Annahme, für die weniger der Ursprung der Sprache als der des Schweigens eine Schwierigkeit bietet, ist besonders von A. MARTY bekämpft worden [10]).

MARTY zufolge widerspricht STEINTHALs Behauptung, dass für *jede* Anschauung, für *jede* Wahrnehmung ein onomatopoetischer Laut zu finden sei, vollkommen der Erfahrung. STEINTHAL hat das eigentlich später selbst anerkannt, ohne darum die Onomatopoie als Prinzip der Sprachschöpfung aufzugeben. MARTY stellt sogar in Abrede, dass irgendeine Anschauung in uns einen onomatopoetischen Laut instinktiv entstehen lassen kann. Er vertritt demgegenüber die Ansicht, dass alle onomatopoetischen Laute als Ergebnis absichtlicher und gewohnheitsmässiger, aber nicht als ursprüngliche Aeusserungen betrachtet werden müssen.

Ich möchte auf MARTY's scharfsinnige Kritik von STEINTHALs und WUNDTs Anschauungen nicht eingehen, nur betonen, dass es sich sowohl bei MARTY wie bei seinen Gegnern nicht um das Problem des Sprach*ursprungs*, sondern nur um das der Sprach*bildung* handelt, also um eine Hypothese bezüglich der Fortentwicklung der Sprache und nicht — wie sie irrtümlicher Weise gedacht haben — bezüglich ihrer Entstehung. Man kann sogar weiter gehen und behaupten, dass der ganze leidenschaftliche Streit zwischen den Nativisten (W. v. HUMBOLDT, HEYSE, RENAN, LAZARUS, STEINTHAL, WUNDT) und den Empiristen (CONDILLAC, TIEDEMANN, GEIGER, MARTY, MADVIG) sich im wesentlichen auf die vermeintlichen Tendenzen und Faktoren bezieht, die beim *Aufbau* der Sprache mitwirken. So versucht z.B. die sog. Erfindungstheorie die Wahl und Gestaltung der Sprachzeichen der Reflexion, der planmässigen Arbeit und Ueberlegung zuzuschreiben, während die Theorie von MARTY dasselbe durch Annahme einer absichtlichen und planlosen (nicht wahllosen) Arbeit erklären will. Nicht viel anders ist es, wenn REGNAUD [11]) die Sprache durch allmähliche Differenzierung der Schreilaute entstanden denkt, ähnlich wie DARWIN [12]) und SPENCER [13]) die Musik aus den Naturlauten abzuleiten versuchen [14]). REGNAUD trachtet wenigstens einen rudimentären Zustand aufzuweisen, welcher der artikulierten und sinnvollen Sprache vorangegangen sein soll; dabei lässt er jedoch gänzlich ausser Acht, dass eine Aeusserung, die *nichts* mit der Sprache zu tun hat, für ihre Entstehung nicht verantwortlich gemacht werden kann.

Die Unhaltbarkeit der onomatopoetischen Theorie kann von meinem Standpunkt aus unschwer bewiesen werden.

Nimmt man an, dass onomatopoetische Laute jene Aeusserungen darstellen, aus denen die Sprache entstanden ist, dann gibt es zwei Möglichkeiten. Sind diese Laute nur reine Nachahmungen ohne jegliche Intention auf gegenseitige Verständigung, dann können sie genau so wenig als Anfänge der Wortlaute betrachtet werden wie die Nachahmungslaute der Vögel. Hat indessen der Urmensch onomatopoetische Laute als Verständigungsmittel gebraucht, dann stellen diese Laute *Worte* dar, mithin Glieder einer *Sprache*, wenn auch einer primitiven und äusserst beschränkten, da es unmöglich ist, Wahrnehmungsgegenstände, Wünsche, Erlebnisse etc. onomatopoetisch auszudrücken. Eine derartig eingeschränkte Sprache kann man sich nicht vorstellen. Werden also im Urzustand der Menschheit lautliche Aeusserungen ohne die Absicht der Verständigung verwendet,

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[9]) H STEINTHAL, Abriss der Sprachwissenschaft, I, 1871, S. 389, und: Der Ursprung der Sprache, 1877.

[10]) A. MARTY, Ueber den Ursprung der Sprache, 1875, und verschiedene Artikel in der Vierteljahrschr. f. wiss. Philos. Bd. 8 bis 16; abgedruckt im ersten Band seiner „Gesammelten Schriften", 1916.

[11]) P. REGNAUD, Origine et philosophie du langage, 1887.

[12]) CH. DARWIN, The Descent of Man. London 1898.

[13]) H. SPENCER, Essays, London 1858.

[14]) G. RÉVÉSZ, Der Ursprung der Musik. Intern. Zeitschr. f. Ethnographie. Bd. 40. 1941. S. 65.

dann stehen sie jenseits des Prinzips „Sprache", vermögen sie demnach nicht als Vorstadia der Sprache zu gelten; sind sie indessen zum Zweck der Verständigung erfunden und ausgebildet, so sind sie eben Aeusserungen der Sprachfunktion. Ein Mittelding zwischen Sprache und Nichtsprache gibt es nicht. In diesem Sinne bemerkt WUNDT, dass er einen geistigen Zustand undenkbar findet, der reif genug ist, die Sprache zu erfinden, und sie doch nicht besitzt [15]).

D. *Die Gebärdentheorie.*

Einige Forscher, stark beeinflusst in ihren Anschauungen von dem Entwicklungsgedanken, wie z.B. WUNDT und SPENCER, behaupten, dass am Anfang der menschlichen Sprachentwicklung die Gebärdensprache stand, woraus sich die Lautsprache allmählich entwickelte. Diese Anschauung hat man mit dem Problem des Ursprungs der Sprache im Zusammenhang gebracht, ohne bemerkt zu haben, dass man durch die Annahme des Primats der Gebärdensprache die Ursprungsfrage bloss verschoben, aber nicht gelöst hat. Es handelt sich bezüglich dieser nicht mehr um die Entstehung der Lautsprache, sondern um die der Gebärdensprache.

Bei der psychologischen Begründung der Gebärdenhypothese geht man von der Anschauung aus, dass die motorischen Ausdrucksäusserungen vom entwicklungspsychologischen Standpunkt aus eine primitivere Stufe darstellen als die Lautäusserungen. Schon dieser Ausgangspunkt ist anfechtbar. Lebende Wesen geben, falls sie über einen klangerzeugenden Apparat verfügen, von ihren inneren Erregungszuständen genau so durch Klanglaute wie durch Körperbewegungen Kunde. Man denke an das Winseln und Knurren des Hundes, den Schreckruf der Amsel, den Lockruf der Glucke, den Fresston des Affen, das Wutgeschrei des Gänserichs, ferner an die mannigfachen Quetsch- und Knarrlaute, Warn-, Schreck- und Schmerzlaute der verschiedenen Tiere, an die Laute des Wohlbefindens, des Paarungsbedürfnisses usw. Es gibt sogar Tierarten, bei denen der lautliche Ausdruck den motorischen an Bedeutung weit übertrifft, wie etwa die Vögel und Affen. Mit Rücksicht auf die *anatomischen und physiologischen Grundlagen* der Stimmerzeugung hat also der Primat der Gebärdensprache gegenüber der Lautsprache keine Wahrscheinlichkeit.

Dass beim Menschen die physiologischen Vorbedingungen sowohl für die motorischen wie auch für die akustischen Äusserungen erfüllt sind, schliesst natürlich die Möglichkeit nicht aus, dass die Gebärdensprache ein früheres Stadium der Sprachtätigkeit darstellt, als die Lautsprache. Ob das wirklich der Fall gewesen ist, darüber wissen wir nichts. In bezug auf die phylogenetische Entwicklung der Sprache sind wir vollkommen auf Vermutungen, Konstruktionen, Schlussfolgerungen angewiesen, wobei mehr die Phantasie als die Logik waltet. Das einzige Erfahrungsmaterial, das uns zur Verfügung steht und woraus wir mit einiger Wahrscheinlichkeit auf die Anfänge der Sprache schliessen können, sind die Sprachen primitivster Völkerstämme. Wenn es sich nun zeigen liesse, dass *alle* auf sehr niedriger Kulturstufe stehenden Völker entweder ausschiesslich oder vorzugsweise die Gebärdensprache verwenden oder dass ihre Lautsprache nachweisbar sich unter dem ständigen Einfluss der Gebärdensprache entwickelt, dann könnte man diese Ergebnisse der ethnologischen Sprachforschung zur Begründung der Lehre vom Primat der Gebärdensprache heranziehen. Man müsste allerdings voraussetzen, dass die Sprachen — mindestens in ihren Anfängen — von *gleichen* Entwicklungsgesetzen beherrscht waren, folglich dass das, was bei der Sprachentwicklung der Primitiven zu beobachten ist, auch in Hinblick auf unsere Sprachentwicklung gegolten hat. Allein unsere ethnologischen Kenntnisse unterstützen die Lehre vom Primat der Gebärdensprache nicht —, wenigstens soweit wie ich darüber orientiert bin.

Erstens gibt es kein Volk, das sich ausschliesslich einer Gebärdensprache bedient. Die Gebärdensprache scheint allerdings eine sehr verbreitete Sprachart bei Primitiven zu sein;

[15]) W. WUNDT, Logik, I, S. 16 u. 47.

aber ebenso sicher ist es, dass alle diese Völker neben einer Gebärdensprache auch noch eine viel entwickeltere Lautsprache haben. Dass Individuen gleicher Sprachgemeinschaft sich bloss durch Sprachgebärden verständigen, kommt nicht vor; beide Spracharten werden abwechselnd, meistens gleichzeitig, einander unterstützend und ergänzend, verwendet. Die allerprimitivsten Stämme der Erde, wie z.B. die Pygmäen in Südafrika und in Ceylon oder die Hottentotten, verständigen sich durch Lautsprachen, die noch dazu einen ziemlich komplizierten grammatikalischen Bau aufweisen. Die strukturelle Ähnlichkeit der primitiven Sprachen mit der Gebärdensprache bei den Sudanesen und Hottentotten — ein Umstand, den WUNDT in seiner „Elementen der Völkerpsychologie" besonders hervorhebt und in Hinblick auf den er die Gebärdensprache als eine Art Ursprache abzuleiten sucht, — besitzt nicht die geringste Beweiskraft. Eine solche Übereinstimmung würde nur beweisen, dass der Wunsch nach Verständigung und Mitteilung beide Ausdrucksweisen in Anspruch nimmt und dass Laut- und Gebärdensymbole einander wechselseitig fördern; sie würde indessen keineswegs die Annahme unterstützen, dass die *eine* Form des sprachlichen Ausdruckes ursprünglicher ist als die *andere*. Diese Theorie, die auf Allgemeinheit Anspruch erhebt, erfordert, dass diese äussere Verwandtschaft zwischen Laut- und Gebärdensprache bei zahlreichen primitiven Sprachen nachgewiesen wird, ferner dass auch bei den entwickelten Sprachen ihr Ueberbleibsel noch deutlich zum Vorschein kommt. Diese Forderung ist bei jetzt noch nicht erfüllt. Auch der Umstand, dass in der äusserst primitiven Ewe-Sprache (Sudan) [16]) die vorhandenen Gebärdenzeichen an Anschaulichkeit und unmittelbarer Verständlichkeit die Wörter und Satzbildungen übertreffen, besagt nichts für die Ursprünglichkeit der Gebärdensprache gegenüber der Lautsprache. Auch wir können manches durch Gebärden — die bekanntlich von Natur eine engere Beziehung zwischen Bedeutung und Zeichen ermöglichen als die Lautsprache — viel anschaulicher und deutlicher ausdrücken als vermittels der Wortsprache. Man denke an die affektiv fundierten Gebärden oder an die Pantomimik bei theatralischen Darstellungen und kultischen Handlungen. Anschaulichkeit und Deutlichkeit ist nichts mit Ursprünglichkeit zu machen. Andererseits gibt es unzühlige Fälle, in denen die Lautsprache die mitzuteilenden und darzustellenden Ereignisse anschaulicher und unmittelbarer zum Ausdruck bringt als die Gebärdensprache, — soweit sie bei diesen Fällen als Ausdrucksmittel überhaupt, in Anmerkung kommt.

Das uns zur Verfügung stehende ethnologische Material spricht also nicht für die Gebärdenhypothese. Im Gegenteil: es macht es sehr wahrscheinlich, dass die Menschen von Beginn an *beide* Verständigungsformen benützen, ihre Gedanken durch artikulierte Lautkomplexionen und Gebärden ausdrücken. Darauf weist der besonders von CUSHING [17]) hervorgehobene Umstand, dass die beiden Spracharten bei den Primitiven in ihrer Entwicklung *autonom* sind. Weder die Lautsprache noch die Gebärdensprache ist nach ihm als die ursprünglichere zu betrachten. Beide Spracharten sollen unmittelbare Äusserungen des einheitlichen Denkens sein. Diese Auffassung hat CUSHING veranlasst, neben den Wortbegriffen noch Gebärdenbegriffen, manuals concepts, anzunehmen. Beide üben aufeinander einen starken Einfluss aus, so dass zwischen ihnen eine besonders starke Wechselwirkung entsteht.

Über den ersten Anfängen der Sprache wissen wir eigentlich nichts. Haben wir das Bedürfnis, die Entwicklung der Sprache von ihrem Ursprung aus zu rekonstruieren, so treffen wir wohl das Richtigste, wenn wir die beiden Spracharten als unmittelbare Äusserungen des *einheitlichen* Denkens und Vorstellens betrachten, die aus zwei voneinander verschiedenen Quellen entsprungen sind, nämlich aus den Gebärden und aus den Lautbildern.

Auch die Ontogenese der Sprache legt die Annahme der Gleichzeitigkeit der beiden Spracharten nahe. In der ersten Entwicklungsphase des Kindes weist nichts darauf, dass die Ausdrucksbewegungen früher auftreten als die lautlichen Äusserungen. Bei Neuge-

[16]) D. WESTERMANN, Grammatik der Ewe-Sprache. Berlin 1907.
[17]) F. H. CUSHING, Manual Concepts, American Anthopologist, V, p. 291.

borenen trifft man schon in den ersten Tagen ihres Lebens sowohl Lautäusserungen (Schreien und Wimmern) als auch Bewegungen der Extremitäten (Streckungen und Bewegungen der Arme und Beine, Abwendung des Kopfes) als Ausdruck von Lust- und Unlustgefühlen. Die Gebärden und die ersten Lallworte stellen sich ziemlich früh ein und ungefähr zu gleicher Zeit [18]). Die erste Gebärde, nämlich die weisende und zeigende, kommt nicht eher zur Ausführung, als bis das Kind mindestens einige Worte begreift, sich bei ihm das Sprachverständnis einstellt [19]).

Man hat auch den Versuch gemacht, die Erfahrungen an Taubstummen für die Gebärdentheorie nutzbar zu machen. Dieser Versuch ist vollkommen misslungen. Es ist leicht zu zeigen, dass die Harmonie zwischen den Zeichen der natürlichen Gebärdensprache bei den verschiedensten Völkern und den Taubstummen mit dem Primat der Gebärdensprache nichts zu tun hat. Die Übereinstimmung erklärt sich aus der *generellen Form* aller Ausdrucksbewegungen und Gebärden, ihrem konkreten, deskriptiven und nachbildenden Charakter, der für die sämtlichen Gebärdensprachen und die Lautsprache überall begleitenden Gebärde typisch ist und sich in ihnen allen manifestieren muss.

Die Annahme, dass sich die Lautsprache aus der Gebärdensprache entwickelt hat, ist auch darum sehr anfechtbar, weil die Wortsprache mit der natürlichen Gebärdensprache keine phänomenale Ähnlichkeit hat und nur eine geringfügige strukturelle Übereinstimmung aufweist.

Schliesslich kann man noch auf eine psychologische und kulturgeschichtliche Erscheinung hinweisen. Die älteste Form der Schriftsprache, die Piktographie, bezieht sich auf die Lautsprache und nicht auf die Gebärdensprache. Bei der piktographischen Darstellung eines Vorgangs handelt es sich stets um die bildliche Darstellung einer *lautsprachlichen* Mitteilung.

Der Primat der Gebärdensprache lässt sich also weder durch biologische bzw. entwicklungspsychologische Argumente noch durch sprachgeschichtliche Erfahrungen wahrscheinlich machen. Dieser Auffassung liegt meiner Überzeugung nach die unberechtigte Identifizierung der Ausdrucksbewegung mit der Gebärde zugrunde. Man hat dabei ausser Acht gelassen, dass die Ausdrucksbewegung eine durch innere Erregungszustände reflektorisch oder instinktiv ausgelöste Bewegung ist, die keine kommunikative Tendenz hat und somit jeder mitteilenden und bezeichnenden Funktion entbehrt. Demgegenüber stellt die Gebärde ein zielbewusstes, willkürliches Zeichen dar, das den Zweck des Hinweisens, Mitteilens, Bezeichnens, Anzeigens verfolgt. Ausdrucksbewegungen gehören ausschliesslich der triebhaft-affektiven Sphäre an, während für das Zustandekommen der Gebärden Verstand und Zielsetzung, d.h. eine willensmässige Einstellung erforderlich ist. Die Elemente der Gebärdensprache sind nicht emotionale Ausdrucksbewegungen, sondern Gebärden, die als solche schon *sprachbezogen* sind [20]). Wie das Wort nicht früher entstehen konnte als die Lautsprache, ebenso konnte die Gebärde als solche der Gebärdensprache nicht zeitlich vorangehen.

### E. *Die tierpsychologische Theorie.*

Wenn nun aber einmal feststeht, dass die körperlichen Ausdrucksbewegungen und die emotionalen Lautäusserungen weder als Urform noch als Vorstufe der Sprache in Betracht kommen, dann könnten wir eigentlich davon absehen, die Lehre, welche die Sprache aus den *tierischen Lauten* abzuleiten sucht, näher zu diskutieren. Die tierischen Lautäusserungen sind reine Ausdrucksbewegungen der vitalen Sphäre. Demnach haben die Argumente, die wir gegen die Auffassung der menschlichen Ausdrucksbewegungen als Vorstufe der Sprache angeführt haben, Geltung auch in Hinblick auf die Auffassung

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18) W. STERN, Die Kindersprache, 3. Aufl. Leipzig. 1922.

19) R. VUYK, Wijzen en spreken in de ontwikkeling van het kleine kind. Ned. Tijdschr. v. Wijsbeg. en Psychol. 1940.

20) G. RÉVÉSZ, De menschelijke hand. Een psychologische studie. 1941.

der tierischen Lautäusserungen als Vorstufe der Sprache. Wollten wir die tierischen Laute als die ersten Ansätze der Sprache betrachten, so müssten wir erwarten, dass die Sprachen, in erster Reihe die primitivsten Sprachen, Worte besitzen, die tierischen Lauten gleichen. Allein so etwas ist uns nicht bekannt. Ferner ist es bei einer solchen Voraussetzung schwer zu verstehen, warum kein Tier auf der Welt trotz der Mannigfaltigkeit seiner Lautäusserungen (siehe die sog. Wörterbücher der Pferde- und Affen-„Sprachen" bei MÁDAY, GARNER, BOUTAN, KELLOGG, YERKES, LEARNED) und der vollkommenen Entwicklung seiner Stimmorgane (Papagei) und trotz der grössten Mühe seitens der Experimentatoren weder zu einer *eigenen* noch auch zur Übernahme einer *fremden* (menschlichen) Sprache — wenn auch nur in Ansätzen — gekommen ist [21]). Bekanntlich begreifen Tiere, die imstande sind menschliche Sprachlaute nachzuahmen, nicht den Sinn ihrer hervorgebrachten „Wortlaute" und sind völlig unfähig, die erlernten Lautbilder dem Bedürfnis nach zu modifizieren bezw. zu Komplexionen zu verbinden. Nicht das geringste Anzeichen dafür lässt sich bei einwandfreier Beobachtung und vorurteilsloser Interpretation der Phänomene finden. Hätten diese angeblich an der „Schwelle" der Sprache stehenden Tiere auch nur den ersten Schritt zum Sprechen getan, so würden prinzipiell alle Schwierigkeiten aufgehoben sein und es würde sich vielleicht nur um eine Zeitfrage handeln, wann die Tiere sich mit uns unterhalten könnten. Diese Tiere besitzen wohl *Stimm*organe, aber keine *Sprach*organe. Sprachorgane besitzt nur der, welcher die Sprachfunktion bezw. die angeborene Anlage zum Sprechen hat.

Tiere verstehen die menschliche Sprache nicht. Wortverständnis haben, heisst, die Worte in ihrer (symbolischen) *Bedeutung* erfassen können. Diese Fähigkeit besitzt kein Tier. Dass gewisse Tiere, wie Hunde, Katzen, Affen auf gewisse Laute zu dressieren sind, die sie in ihrem Verhalten gegenüber Menschen bestimmen, widerspricht dieser Auffassung nicht. Ein Laut ist ein Zeichen, ein Signal, das eine bestimmte Reaktion oder eine Handlung auszulösen imstande ist. Ob ich den Hund darauf dressiere, auf das *Wort* „geh" oder auf den *Satz* „geh hinaus" oder auf einen *Pfiff* oder auf einen gezeigten *Buchstabe* oder auf das Erscheinen der roten *Farbe* hin das Zimmer zu verlassen, ist gleichgültig. Das Dressurwort „geh" ist für den Hund kein Wort und auch das phonetische Gebilde „geh hinaus" bildet für ihn keinen Satz; in gleicher Weise ist das optische Zeichen „A" für ihn kein Buchstabe und rote Farbe keine Farbe im Sinne dieses Wortes. Alles das sind gleichwertige und Gleiches bedeutende Signale, sinnlich wahrnehmbare Zeichen, auf die das Tier dressiert werden kann.

Nach alledem sind wir berechtigt die tierpsychologische Hypothese für das Ursprungsproblem, ja, für die vergleichende und genetische Sprachwissenschaft aus der Diskussion endgültig auszuschalten [22]).

F. *Ontogenetische Theorie.*

Die Frage nach dem Ursprung der Sprache lässt sich auch nicht durch den Hinweis auf die Anfänge der *Kindersprache* beantworten. Wenn man unter der Vorstufe der Sprache solche Lautäusserungen versteht, die bereits gewisse *wesentliche* Elemente der Sprache in sich schliessen und im Grunde genommen von denselben Tendenzen bestimmt sind wie die Sprache, dann lassen sich die Lalllaute *nicht* als Vorstufe der *Sprache* betrachten, sondern vielmehr als Vorstufe des *Sprechens*. Die Lallperiode, die spontane Einübung der Sprechtätigkeit, die dem Sprechen vorangeht, hat ausschliesslich eine

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[21]) Siehe meine Ausführungen über die autochthone „Tiersprache" und über das hypostasierte Wortverständnis der Tiere in meiner bereits zitierten Abhandlung: „Die menschlichen Kommunikationsformen und die sog. Tiersprache. Vgl. Nederl. Akad. van Wetenschappen. Proceedings Vol. 43. 1940.

[22]) Durch analoge Beweisführung lässt sich auch die irrtümliche Auffassung bestreiten, dass der sog. tierische Gesang das Urmodell der Musik sei. Vergl. meine oben zitierte Abhandlung über den Ursprung der Musik.

physiologische Bedeutung. Sie kann nicht als eine Stufe der inneren Sprachentwicklung gelten. Wenn auch die Sprechtätigkeit das Lallen voraussetzt, so liegt doch der Sprache eine wesentlich andere Funktion zugrunde wie dem Lallen. Es ist unzweifelhaft, dass schon in der vorsprachlichen Periode des Kindes die Disposition zum Sprechen vorliegt, die in ihrer Entfaltung und Richtung von der Sprache der Umgebung bestimmt wird. Es ist sogar möglich, ja sehr wahrscheinlich, dass in dieser Periode die Sprachfunktion ihre Wirksamkeit bereits innerlich, gleichsam unterirdisch vorbereitet, obschon sie sich erst später aktualisiert. Äussert sie sich aber einmal, dann tut sie es in einer Form, die *dem Sinn der Sprache* entspricht.

Das neugeborene Kind kann nicht sprechen, weil es in jeder Hinsicht, in organologischer wie in intellektueller, unfertig ist. Der freilich nur in unserer Phantasie existierende Urmensch war weder organologisch noch intellektuell unfertig. Er ist daher mit dem Kind nicht zu vergleichen. Täte man dies, so müsste man annehmen, dass die ersten sprachähnlichen Äusserungen des Menschen das Lallen gewesen seien, dies ist aber eine Annahme, die geradezu sinnlos ist. Der Urmensch war kein Säugling und wenn er etwa dennoch nicht hat sprechen können, dann müsste das eine andere Ursache gehabt haben wie beim Kind. War indessen der hypostasierte sprachlose „Urmensch" schon für das Sprechen prädisponiert, dann ist es nicht zu verstehen, warum er der Sprache noch nicht fähig war. Fehlte ihm aber der innere Sprachsinn, dann war er noch kein Mensch, weil ihm eben das Wesentliche fehlte, das den Menschen von allen übrigen Lebewesen unterscheidet, nähmlich die Kontaktform der Sprache.

Wenn man die ersten Etappen der sprachlichen Entwicklung des menschlichen Individuums mit den Anfängen der Sprache überhaupt vergleicht, darf man eben nicht ausseracht lassen, dass die Sprachfunktion nicht nur im Gebrauch der Laut- bezw. Gebärdensprache, sondern vor allem im *Sprachverständnis* zum Ausdruck kommt. Wenn ein kleines Kind die Muttersprache versteht und sich in seinem Tun und Lassen von der Sprache der Umgebung beeinflussen lässt, so hat es die erste *Sprachstufe* bereits erreicht, wenn auch das aktive Sprechen erst später nachkommt. Auch uns Erwachsenen ist das Geheimnis einer neuen Sprache bereits enthüllt, sobald wir sie verstehen. Ist Sprachverständnis einmal vorhanden, dann ist auch der Weg zur Sprachtätigkeit offen und die Aktionsfähigkeit des Denkens bereits gesichert.

**Biochemistry.** — *Behaviour of microscopic bodies consisting of biocolloid systems and suspended in an aqueous medium. VI. Composition of degenerated hollow-spheres, formed from complex coacervate drops (gelatine-gum arabic).* By H. G. BUNGENBERG DE JONG and E. G. HOSKAM. (Communicated by Prof. H. R. KRUYT.)

(Communicated at the meeting of January 31, 1942.)

*Introduction.*

In Communication IV of this series we described the vacuolization phenomena of complex coacervate drops, when they are brought in contact with dist. water [1]). It appeared that when the coacervate drops are originally charged negatively, the primary vacuolization passes into foam formation and finally results in hollow spheres. It is supposed that abnormal osmosis is the cause of the formation of this foam structure and of the hollow spheres. Further investigation gives strong support to this supposition [2]). Nevertheless we felt the necessity of knowing something about the composition of the coacervate skin which forms the wall of the hollow spheres, three questions especially requiring a solution:

1. Does the wall of the hollow spheres still consist of complex coacervate, or, owing to the removal of one of the colloid components (gum arabic), does it consist of the second colloid component only (gelatine?)
2. Is it possible to foresee any changes in composition which may arise with regard to the original composition?
3. Is the composition of the liquid wall such as we can expect for a complex coacervate of still negative charge?

In what follows it will be seen that the results of the chemical analysis support the views concerning the mechanism of the formation of hollow spheres, developed previously. In these theories we had started from the supposition that the wall of the hollow spheres still consists of a coacervate with a negative charge.

*Experimental.*

On account of their large vacuoles the hollow spheres themselves are no suitable objects for analysis, at least if we wish to find out the water percentage of the wall besides the gelatine-gum arabic proportion.

So we must content ourselves with analyzing the coacervate drops free from vacuoles formed in consequence of spontaneous degeneration. The circumstances determine the length of existence of the hollow spheres, in the experiments described here they lasted at most 20—30 minutes. So long as the hollow spheres are typical, i.e. so long as they have a very thin wall, they will settle only very slowly in a sedimentation tube. As the vacuole volume decreases they settle more rapidly. The sedimentation layer forming in a sedimentation tube when it is left undisturbed, consists therefore mainly of coacervate drops free from vacuoles or containing a little vacuole only. Here another difficulty arises: whereas the ordinary coacervate drops easily coalesce on sedimentation to a clear coacervate layer, this is not the case with "degenerated" hollow spheres. Although they too are liquid internally, their surface is apparently in a particular condition, owing to

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[1]) H. G. BUNGENBERG DE JONG and O. BANK, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **42**, 274 (1939).

[2]) H. G. BUNGENBERG DE JONG, O. BANK and E. G. HOSKAM, Protoplasma **34**, 30 (1940).

which coalescence practically does not or hardly takes place. But they flatten each other considerably in the sediment layer [1]), so that only little of the medium liquid is enclosed.

In consequence of this difficulty, analysis can at most give a slightly too high figure for the water percentage of the coacervate, but this will have no practical effect on the proportion of gelatine and gum arabic as the small quantity of medium liquid enclosed contains relatively few colloids.

For the calculation of the composition of coacervate and the above liquid it was necessary to determine the dryweight and the nitrogen percentage. The dryweight was determined as follows: a weighed quantity of the sample was placed for one hour in a nickle box on a boiling water bath and then for one hour in an electric drying stove at 120° C.

The nitrogen was determined by DEKKER's method [2]). With the aid of the dry weights determined thus and the N-percentages of the dry substance and using the N-percentages of the gelatine (determined at 16.65 %) and gum arabic (= 0.33 %) the gelatine and gum arabic percentage is calculated [3]).

*Results.*

We started from a system which is known to give excellent hollow spheres. For this isohydric solutions were prepared (pH 3.67) of gelatine and gum arabic from the corresponding stock solutions.

Stock solutions: 22 g air-dry gelatine resp. gum arabic are dissolved in 380 g dist. water.

I. Isohydric gelatine solution: 40 ccm stock sol. + 13 ccm 0.1 N HCl + 47 ccm $H_2O$ (dryweight determination 1.985 %).

II. Isohydric solution of gum arabic: 40 ccm stock solution + 4.5 ccm 0.1 N HCl + 55.5 ccm $H_2O$ (dryweight determination 2.06 %).

In each of 4 sedimentation tubes with a contents of 250 ccm we then placed: 84 ccm sol. I (isohydric gelatine) and 166 ccm sol. II (isohydric gum arabic).

We always worked at a temperature of 40° C. After sufficient sedimentation 2 tubes were placed for ca. 10 minutes in cold water and after gelatination of the coacervate the upper layer was poured off and replaced by 250 ccm of an isohydric HCl solution [4]).

These two tubes were again placed in the thermostat and after the contents had reached the right temperature they were well shaken. Typical hollow spheres then formed which slowly sank. The other two tubes did not undergo this treatment. Dry weight and N percentage were then determined of each of the 4 tubes of the upper layer as well as of the coacervate, resp. of the layer of degenerated hollow spheres. With the aid of these values and with the dryweight and nitrogen percentage of solutions I and II we then calculated the gelatine and gum arabic percentages in each of the layers.

In the following table are given the analysis data and the results calculated from them for the original coacervated system (left) and for the system after passing the stage of the hollow spheres (right).

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1) When the sediment tube is placed in cold water the complex coacervate gelatinises in a short time (within 20 minutes). In the case of an ordinary complex coacervate the gelatinized sediment layer forms a cohering mass (turbid owing to vacuolization). In the case of a sediment layer of degenerated hollow spheres this layer can be separated by vigorous shaking to a suspension of separate polygonal bodies. (The coacervate drops mentioned before, they are flattened by contact with each other, but have not coalesced.)

2) W. A. L. DEKKER, Handleiding voor het klinisch chemisch onderzoek, 3e dr. Leiden 1940.

3) See Kolloid Beihefte **43**, 215 (1936).

4) Prepared by adding the calculated quantity of HCl to distilled water.

TABLE.

| | Original system | | After passing the hollow spheres stage | |
|---|---|---|---|---|
| | Coacervate layer | Equilibrium liquid | Sedimentation layer | Upper layer |
| Dryweight % | 12.93 | 0.795 | 17.16 | 0.617 |
| N-percentage of dry substance | 7.21 | 3.59 | 7.66 | 5.21 |
| Gum arabic (A) % | 7.48 | 0.64 | 9.45 | 0.43 |
| Gelatine (G) % | 5.45 | 0.16 | 7.71 | 0.18 |
| A/G | 1.37 | 4.0 | 1.23 | 2.38 |

*Discussion.*

The results enable us to answer the questions asked in the introduction:

1. The degenerated hollow spheres do indeed contain gum arabic besides gelatine. So the wall of the hollow spheres does not exist exclusively of gelatine, but of a typical complex coacervate.

2. The composition of the degenerated hollow spheres is changed in two respects with regard to the original composition:

*a.* The water percentage is smaller (dryweight of 12.93 % has increased to 17.16 %).

*b.* The proportion of the two colloids has shifted in favour of the gelatine, which also applies to the upper layer (see lowest horizontal row in the table).

As regards *a.* this change is to be expected from the removal of neutral salt ($CaCl_2$) formed from the counter ions of the two colloids (Ca from gum arabic, Cl′ from the gelatine). On complex coacervation the two colloid ions + water combine in principle to the coacervate, the remaining neutral salt dividing itself over the two liquid layers. Neutral salts increase the waterpercentage of the complex coacervates and consequently the removal of the upper layer and its substitution by an isohydric HCl solution results in the decrease of the waterpercentage of the complex coacervate. It is also the cause of the primary vacuolization, which on sufficiently negative coacervates passes secondarily into a foam structure and the formation of hollow spheres.

As regards *b.* we should remember that the mixing proportion chosen of the sols is such that a negatively charged complex coacervate is formed. With the pH given there is then an excess of gum arabic (A) in the total system from the point of view of the mutual charge compensation of the two colloids of opposite charges. This excess of gum arabic is divided over coacervate and equilibrium liquid in such a way that A/G in the coacervate is smaller than in the total system, while A/G in the equilibrium liquid is greater than in the total system [1]).

Of a coacervate with positive charge the reverse is true while on charge compensation these proportions become mutually equal:

| | | | | | |
|---|---|---|---|---|---|
| Coacerv. neg. | (A/G) coac. | < | (A/G) total | < | (A/G) equil. liquid. |
| Uncharged coac. | (A/G) coac. | = | (A/G) total | = | (A/G) equil. liquid. |
| Coac. pos. | (A/G) coac. | > | (A/G) total | > | (A/G) equil. liquid. |

[1]) H. G. BUNGENBERG DE JONG, Kolloid Beihefte 43, 213 (1936). C.f. fig. 11, p. 234, loc. cit., where this appears from the curves for $\frac{A}{A+G}$, which applies therefore also for $\frac{A}{G}$.

That this is indeed true of the original coacervate is seen when A/G in the total system is calculated. From the dryweight of the two stocksols (G = 1.985 %; A = 2.06 % and the mixing proportion (84 cc gelatine sol + 166 cc gum arabic sol) we calculate a percentage of 0.667 % gelatine and 1.368 % gum arabic, i.e. for the total system A/G = 2.05. This figure lies indeed between the two values for A/G, viz. 1.37 and 4.0.

When in preparing the hollow spheres we remove the original equilibrium liquid (which has a comparatively high gum arabic percentage), replacing it by an isohydric HCl solution, the gum arabic still present in the coacervate will again divide over the two layers, the consequence of which will be a decrease of the A/G proportion in the coacervate, which is indeed proved by the table (1.37 → 1.23).

3. The question if the complex coacervate forming the wall of the hollow spheres has the composition typical of a negative coacervate can now at once be answered in view of what has been said above. This is already indicated by the fact that A/G of the upper layer (2.38) is greater than A/G of the sedimentation layer (1.23). Moreover we can see if A/G in the total system, as it has been formed by the removal of the upper layer, lies indeed between these two values. Auxiliary determinations on a smaller scale showed that the original coacervate volume was 22.4 ccm (1.12 cc for 12.5 cc final volume, this amount was here taken 20 times).

By removing the upper layer = 250 — 22.4 = 227.6 cc we subtracted from the total system: 227.6 × 0.0064 = 1.46 g gum arabic and 227.6 × 0.0016 = 0.36 g gelatine, while originally there was 250 × 0.01368 = 3.42 g gum arabic and 250 × 0.00667 = 1.67 g gelatine. So in the system there was left 1.96 g gum arabic and 1.31 g gelatine, from which it follows that for the total system A/G = 1.50, which value is indeed between the A/G values of 1.23 and 2.38 in the way characteristic of a negative complex coacervate.

*Summary.*

1. The composition of degenerated hollow spheres formed from the complex coacervate gelatine-gum arabic is investigated.
2. Besides gelatine they contain gum arabic and are therefore still complex coacervates.
3. Their water percentage is lower than that of the original coacervate and they contain relatively less gum arabic.
4. The modifications in 3 can be foreseen from the treatment the original coacervate has undergone.
5. From the analysis figures it can be concluded that the degenerated hollow spheres are complex coacervates with negative charge.
6. What is said in 5 is in accordance with the views concerning the mechanism of the formation of hollow spheres published elsewhere.

*Leiden, Laboratory for Medical Chemistry.*

**Biochemistry.** — *Tissues of prismatic celloidin cells containing Biocolloids.* VII. *Stagnation effects.* By H. G. BUNGENBERG DE JONG and B. KOK. (Communicated by Prof. H. R. KRUYT.)

(Communicated at the meeting of January 31, 1942.)

In communication V of this series the effects were studied on the complexcoacervate gelatine + gum arabic of a number of salts and non-electrolytes added to the 0.01 N acetic acid [1]). The effects occurring when the new medium is led continuously past the membrane (inflow effects) have been described in that communication, likewise the effects occurring when after that 0.01 N acetic acid is continuously led past the membrane (outflow effects). In some substances added to the 0.01 N acetic acid it was seen that some special effects are obtained, when the tap of the reservoir containing the inflowing liquid is closed. As these effects are only the consequence of the stagnation of the liquid flowing past the membrane, they were called *stagnation effects*. They have been observed in 5/9 mol glucose, saccharose, glycerine and 20 m. aeq. KCl; but the interpretation which we shall give below makes us expect these effects to be far more general. That we cannot further observe them is possibly owing to the fact that the in- and outflow effects often take place with such a rapidity and intensity, as to render the observation of the comparatively weak stagnation effects very difficult.

We will here give a short description of the stagnation effects with 20 m. aeq. KCl. After the coacervation with 0.01 N acetic acid has been brought about and the parietal coacervate contains only few little vacuoles, we change to 0.01 N acetic acid + 20 m. aeq. KCl. We then see the little vacuoles left in the parietal coacervate disappear and when we continue to lead this medium past the membrane the result is the inflow vacuolization described in communication V. We do not, however, wait for this result, but turn off the tap of the reservoir, then we see a *new vacuolization arise* (many little vacuoles) [2]), which decreases when the tap is quickly turned on again [3]). This *stagnation effect* consisting of vacuolization which decreases when the medium liquid is made to flow again, can be repeated a few times, but after some time the effect becomes less intensive finally not to occur at all. So the *stagnation effect* is a phenomenon occurring only at the beginning of the inflow period and is apparently connected with the fact that medium and contents of the cells are not yet in equilibrium. Figure 1 illustrates our interpretation of the *stagnation effect*: in this figure the object glass against which the celloidin membrane is located is shaded and the border between membrane and the adjacent medium flowing past it is indicated by a vertical line. Apart from the lumen of the cell the space in between contains 1. celloidin wall on the right, 2. celloidin wall on the left and 3. stagnating liquid space between membrane and object glass. But as these are also accessible to the diffusing substance these details have no fundamental significance for us and therefore we have indicated all this as cell in the figure.

We now set out on the ordinate the concentration of the substance added to the medium. In the medium liquid flowing rapidly past the membrane this concentration may be taken as constant and is therefore represented by a horizontal line from a, the celloidin wall, to the right. A short time after the onset of the inflow there is in the cell a certain quantity of

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1) H. G. BUNGENBERG DE JONG and B. KOK, Proc. Ned. Akad. v. Wetensch., Amsterdam, **45**, 67 (1942).

2) Any vacuoles that may still be present then disappear rapidly.

3) Simultaneously with the decrease of the vacuoles formed on stagnation a new generation of vacuoles belonging to the normal inflow effect may be formed.

the diffusing substance, the concentration of which decreases rapidly to the left of the celloidin wall. The course of this concentration is indicated in fig. 1 by straight lines for the sake of simplification.

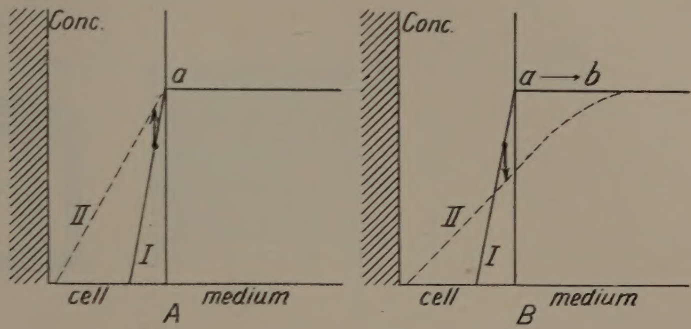


Fig. 1.

Incidentally we note that this consideration gives us at once an explanation of the detail mentioned in Communication V, that a new vacuolization always begins deep in the coacervate, i.e. on that side of the coacervate which is turned to the medium liquid flowing past the membrane. For this zone of the cell always comes first into contact with the diffusing substance coming from the medium. Moreover the concentration change takes place much more rapidly here than more to the left in the cell, which always promotes the occurrence of vacuolization.

For the explanation of the stagnation effect we now compare A and B in fig. 1. We suppose that line II in A shows the situation some time later. It is clear that at any depth in the cell the concentration of the substance diffusing inwards only increases in the course of time from the onset of the inflow until equilibrium has been reached.

In figure 1 B we shall now see what happens when we stop the flow of the medium liquid. Now there is no longer a liquid flowing past the membrane which preserves the concentration of the diffusing substance at a constant value, practically to the celloidin wall (a). As the diffusion continues, a certain zone of the stagnant medium (a—b) becomes poorer in diffusing substance, so that after some time the concentration course will be represented by line II in B. It is seen that in a zone of the cell lying close against the celloidin membrane, the concentration of the substance diffusing inwards has not increased, but decreased (compare the two arrows in fig. 1 A and B). Thus we arrive at the conclusion that *stagnation effects* are practically *local outflow effects,* with which the fact is in accordance that with glucose resp. 20 m. aeq. KCl vigorous and rapid vacuolization takes place on outflow. It also stands to reason that stagnation effects can be well observed exactly after a short period of inflow, that they are less distinct after longer inflow and finally do not occur at all after a sufficiently long period of inflow. Finally it is also clear that as there is here only a temporary local and not very great decrease of the concentration, the vacuolization is only weak in the stagnation effect and therefore possibly escapes observation in less favourable substances than KCl and glucose. Stagnation effects on outflow have not yet been observed by us. They ought also to occur, but possibly the circumstances are even less favourable here than on inflow.

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